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On a Queueing System with Processing of Service Items under Vacation and N-policy with Impatient Customers

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Abstract: We assume that customers arrive at a single server queueing system according to Markovian Arrival process. When the system is empty, the server goes for vacation and produces inventory for future use during this period. The maximum inventory level permitted is L. The inventory processing time follows phase type distribution. The server returns from vacation when there are N customers in the system. Service time follows two distinct phase type distributions depending on whether the processed item is available or not at service commencement epoch. Customers join the queue with probability p and balk with probability 1-p. Also customers waiting for service become impatient and renege after a random time period which is exponentially distributed. We find the distribution of time until the number of customers hit N. Several system performance characteristics are computed. Also we computed Laplace Stieltjes Transform of the waiting time distribution for the case of no reneging. For the special case of no reneging, some numerical experiments for computing individual optimal strategy, maximum revenue to the server and social optimal strategy are also discussed.

Keywords: Balking and reneging of customers, individual and social optimal strategy, L- and N- policies, Laplace-Stieltjes transform, Markovian arrival process, phase type service and inventory processing, server vacation to produce inventory, waiting time distribution.

1. Introduction

In vacation queueing systems, the server may not be available for a period of time due to working on some supplementary jobs, doing some maintenance work or server's failure that interrupt customer service or simply taking a break. Levy and Yechiali [11] introduced the concept of server vacaction. Doshi [5] gives an excellent review of work done in vacation queueing models until 1985. Further developments could be accessed from Takagi [13] and Tian and Zhang [14].

In most of the work reported in queueing theory it is implicitly assumed that if the

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server is ready to serve and the customers are available to receive service then the service process proceeds. Either availability of "additional" items required to provide service is not taken into consideration/ignored or its abundance is taken for granted. In the latter case the holding cost incurred is completely ignored. Sometimes the item(s) required for service maynot be available. In such cases service cannot be provided even when the server is readily available and customer(s) are waiting. Typical example in medical case is operation theatre. In the absence of 'stent' for a heart patient in need of it, surgery cannot be performed. In a vehicle repair shop a vehicle requiring a specific part replacement, cannot be serviced if spares are not available.
(Expansions of various abbreviations are given in 'Notations and abbreviations' below the introduction.)

Thus in several service systems, availability of both customers and servers cannot guarantee service. This will be explained in detail later. This naturally leads to the investigation of availability of additional item(s) required to provide service. As a consequence some optimization problems arise– how much of additional item(s) to be processed at a stretch (optimal value of L) and what should be the optimal value of N? This leads to the consideration of holding cost, shortage cost and associated revenue loss. Kazimirisky [10] seems to be the first to introduce 'additional items needed for service'. He considered a single server queue with Batch Markovian Arrival and General Service time distribution denoted as BMAP/G/1 queue, with the server engaged in producing additional items whenever the customers are not waiting. Exactly one processed item is required for each customer. The customer service time distribution depends on whether processed item is available or not. Thus there are two distinct service time distributions.

Baek *et al.* [2] considered MMAP (Marked Markovian Arrival Process) of customers of two types– type I (high priority) and type II (low priority). Both type of customers require a certain minimum number of additional items to start their service. Type I customers do not have a waiting space. If a type I customer is in service while another type I customer arrives, the latter leaves the system. On the other hand if a type II customer is in service, the former is pushed out of the system by the type I arrival provided the number of additional items available is at least equal to the minimum number required to start its service. Else, it leaves the system without changing the status. Type II customers have an infinite capacity waiting space. Additional items arrive to the system according to MAP. Dhanya *et al.* [3] extend the above to retrial queueing set up.

Hanukov *et al.* [8] analyze a simple queueing system where again additional items are needed for service of a customer (one item for each customer). Arrival process is Poisson and service time is exponentially distributed. Accumulation of N customers (N-policy) in queue to commence service, once server becomes idle, was introduced by Yadin and Naor [18]. This has the advantage that the length of a busy period becomes larger when server is activated on accumulation of N or more customers, thereby bringing down the expected cost incurred per unit time. Divya *et al.* [4] considered single server queue in which the customers arrive according to Markovian Arrival Process. When the system is empty, the server goes for vacation and produces inventory for future use during this period. The maximum inventory level permitted is L. The inventory processing time follows phase type distribution [12]. The server returns from vacation when there are N customers in the system. The service time follows two distinct phase type distributions depending on whether the processed item is available or not at service commencement epoch. They analysed the distribution of time until the number of customers hit N or the inventory level reaches L, distribution of idle time, the distribution of time until the number of customers hit N and also the distribution of number of inventory processed before the arrival of first customer. They also provided Laplace Stieltjes Transform of busy cycles in which no item is left in the inventory and atleast one item is left in the inventory. They performed some numerical computations to evaluate the expected idle time, standard deviation and coefficient of variation of idle time of the server.

In real life, people become impatient while waiting for service. Hence to model reality, we should take into consideration customers' impatience. To characterize customers' impatience, some terminologies like balking, reneging and retrials are employed in queueing system. Balking customers decide not to join the queue if it is too long and reneging customers leave the queue if they have waited too long for service. Retrial queues study systems where the customers do not wait in a line when server is found to be busy; instead they keep repeating their attempts to access the server at random time points (see Falin and Templeton [6], Artalejo and Gomaz-Corral [1]). Wang *et al.* [16] has presented a review on queueing systems with impatient customers.

Yechiali [19] considers an M/M/c queue, where $c=1$, or $1<c<\infty$, or $c=\infty$. At random epochs, the system as a whole gets shock and as a consequence it breaks down. This results in the loss of all customers who are either on the wait for service or undergoing service. When the system is down and undergoing repair process, newly arriving customers become impatient. Their impatience is reflected through a timer of random duration–each customer has his own timer. If a timer expires before the system is repaired, that customer abandons the queue. He analyzed the above described model and derived various quality of service measures such as mean sojourn time of a served customer; proportion of customers served; rate of lost customers due to disasters; and rate of abandonments due to impatience.

Wang and Zhang [17] consider a single-server service-inventory system where customers arrive according to a Poisson process and service times are independent and exponentially distributed. A customer takes exactly one item from the inventory upon service completion. A continuous review policy is adopted to replenish the inventory. With two different information levels, i.e. the fully unobservable case and the partially observable case, arriving customers decide whether to join or to balk the system. They investigated customers individually optimal and socially optimal strategies, and further consider the optimal pricing issue that maximises the servers revenue. Some numerical experiments were carried out to show that the individually optimal joining probability (or threshold) is not always greater than that of socially optimal one. It was observed that, to maximise the servers revenue, concealing some system information from customers may be more profitable. Conversely, to maximise the social welfare, customers need more system information. Finally, numerical results in the fully unobservable case illustrate a reasonable phenomenon that the revenue maximum is equal to social optimum in most cases.

The present paper extends the work of Hanukov *et al.* [8] to a single server, Markovian Arrival, Phase Type distributed processing time and Phase Type distributed service time (Two distinct types-one when the processed item is available and the other when processed is not available)which we denote as MAP/(PH,PH)/1 queue under N -policy with requirement of one additional item to each customer. At service completion epoch, if the server becomes idle for want of customers, he starts processing additional items required to serve customers. A maximum of L items are processed at a stretch. However during this process, if number of customers in the system hits N, then the server immediately commences service. Processing time of items have Phase type distribution. Service time also follows Phase type distribution; however two distinct distributions are considered–one when an additional item is available to start service and the other when no additional item is available. This is already discussed in Divya *et al.* [4]. In this paper we extend that to the case where the customers are impatient. In addition we investigate the individual, social and system optimal strategies by introducing appropriate costs associated with certain system parameters.

Next we turn to further details of the present work. We consider a single server queueing system in which the customers arrive according to Markovian Arrival process. When the system is empty, the server goes for vacation and produces inventory for future use during this period. The maximum inventory level is L. The inventory processing time follows phase type distribution. The server returns from vacation when there are N customers in the system. Service time follows two distinct phase type distributions according as there is no processed item or there are processed items at the beginning of service. Customers join the queue with probability p and balk with probability $1-p$. Also the customers waiting for service may become impatient and renege after a random time period which is exponentially distributed. Whereas Wang and Zhang [17] follow replenishment policy through external sources in the context of queueing-inventory, we investigate the system in which the item is processed by the server himself. Further, in Wang and Zhang model, the server has to stay idle when inventory level drops to zero; in the present model the server processes the item and serves customer if at a service commencement epoch the item is not available.

In all papers cited above that deal with additional items for service/inventory /processed item, it is assumed that service time duration for a customer in the presence of additional items are shorter than that starts in the absence of such an item. Thus the availability of such an item reduces the waiting time of customers in the system. This additional items are in some cases reusable. In this paper we consider the case of non-reusable items.

The rest of the paper is arranged as follows. The model description and mathematical formulation are given in section 2. Section 3 provides steady state analysis of the model, the distribution of time until the number of customers hit N and some other performance measures. For the special case of no reneging some numerical experiments for computing individual optimal strategy, maximum revenue to the server and social optimal strategy are discussed in section 4. Section 5 contains a special case in which the system is working in normal mode.

Notations and abbreviations used in the sequel:

- $\boldsymbol{e}(a)$ = Column vector of 1s' of order a.
- $\boldsymbol{e}'_a$: Transpose of $\boldsymbol{e}_a$.
- $\boldsymbol{e}$ = Column vector of 1s' of appropriate order.
- $CTMC$: Continuous time Markov chain.
- I_a = identity matrix of order a.
- $\boldsymbol{e}_a(b)$ = column vector of order b with 1 in the ath position and the remaining entries zero.
- PH : Phase Type
- MAP : Markovian Arrival Process
- LST : Laplace-Steiltjes Transform
- $LDQBD$: Level Dependent Quasi-Birth and-Death
- $d_{ij}^{(k)}$: entries of D_k, $k = 0$ or 1
- δ_l : l^{th} entry of $D_1\boldsymbol{e}$.

Highlights of this paper are:

- Extends Hanukov *et al.* [8] to the case of Markovian arrival process. In addition, we set an upper bound (L) on the maximum number items processed by the server at a stretch. Further we introduce the N-policy for the number of customers to accumulate, in order to start a fresh service cycle.

- The inventory processing time and service time distributions follow more general distributions (phase type).
- Extends Divya *et.al.* [4] by introducing the two types of customer impatience: balking and reneging. Thus the system turns out to be a level dependent QBD.
- For the special case of no reneging, some numerical experiments to compute individual optimal strategy, maximum revenue to the server and social optimal strategy are also discussed.

2. Model Description and Mathematical formulation

We assume that the customers arrive at a single server queueing system according to MAP with representation (D_0, D_1) of order n. At the end of a service if the system is left with no customer, the server goes for vacation and produces inventory for future use during this period. The maximum inventory level is restricted to L. Processing time for each item in the inventory follows phase type distribution $PH(\alpha, T)$ of order m_1. Server returns from vacation when there are N customers in the system. Service time follows $PH(\beta, S)$ of order m_2 when there is no processed item and it follows $PH(\gamma, U)$ of order m_3 when there are processed items. Customers join the queue with probability p and balk with probability $1-p$. Also the customers waiting for service may become impatient and renege after a random time period which is exponentially distributed with parameter $(n-1)\phi,\ n \ge 1$, where n is the number of customers in the system.

Let $Q^* = D_0 + D_1$ be the generator matrix of arrival process and $\boldsymbol{\pi}^*$ be its stationary probability vector. Hence $\boldsymbol{\pi}^*$ is the unique (positive) probability vector satisfying

$$\boldsymbol{\pi}^* Q^* = 0,\ \boldsymbol{\pi}^* \boldsymbol{e} = 1.$$

The constant $\beta^* = \boldsymbol{\pi}^* D_1 \boldsymbol{e}$, referred to as *fundemental rate*, gives the expected number of arrivals per unit of time in the stationary version of the MAP. It is assumed that arrival process is independent of the inventory processing and service process.

The model described in above can be studied as a level dependent quasi-birth-and-death (LDQBD) process. First we introduce the following notations:

At time t:

$N(t)$: the number of customers in the system at time t,

$I(t)$: the number of processed inventory,

$$J(t) = \begin{cases} 0, & \textit{when the server is on vacation}, \\ 1, & \textit{when the server is busy serving a customer}. \end{cases}$$

$K(t)$: the phase of the inventory processing/service process,

$M(t)$: the phase of arrival of customer.

It is easy to verify that $\{(N(t), I(t), J(t), K(t), M(t)) : t \geq 0\}$ is a LDQBD with state space

$l(0) = \{(0, i, 0, k_1, l) : 0 \leq i \leq L-1, 1 \leq k_1 \leq m_1, 1 \leq l \leq n\} \cup \{(0, L, 0, l) : 1 \leq l \leq n\}.$

For $1 \leq h \leq N-1$,

$l(h) = \{(h, i, 0, k_1, l) : 0 \leq i \leq L-1, 1 \leq k_1 \leq m_1, 1 \leq l \leq n\} \cup \{(h, L, 0, l) : 1 \leq l \leq n\}$

$\cup \{(h, 0, 1, k_2, l) : 1 \leq k_2 \leq m_2, 1 \leq l \leq n\} \cup \{(h, i, 1, k_3, l) : 1 \leq i \leq L, 1 \leq k_3 \leq m_3, 1 \leq l \leq n\}$

and for $h \geq N$,

$l(h) = \{(h, 0, 1, k_2, l) : 1 \leq k_2 \leq m_2, 1 \leq l \leq n\} \cup \{(h, i, 1, k_3, l) : 1 \leq i \leq L, 1 \leq k_3 \leq m_3, 1 \leq l \leq n\}.$

Note that when $0 \leq N(t) \leq N-1$ and $I(t) = L$, the server will be idle as the inventory level reaches its maximum level 'L' and the number of customers doesnot hit 'N'. So $K(t)$ neednot be considered.

The infinitesimal generator of this CTMC is

$$Q_1 = \begin{bmatrix} B_0 & C_0 & & & & & & & \\ B_1 & E_1 & I \otimes pD_1 & & & & & & \\ & B_2 & E_2 & I \otimes pD_1 & & & & & \\ & & \ddots & \ddots & \ddots & & & & \\ & & & B_{N-2} & E_{N-2} & I \otimes pD_1 & & & \\ & & & & B_{N-1} & E_{N-1} & F & & \\ & & & & & B'_N & A_1^{(N)} & A_0^{(N)} & \\ & & & & & & A_2^{(N+1)} & A_1^{(N+1)} & A_0^{(N+1)} \\ & & & & & & \ddots & \ddots & \ddots \end{bmatrix}.$$

The boundary blocks B_0, C_0, B_1 are of orders $(Lm_1+1)n \times (Lm_1+1)n$, $(Lm_1+1)n \times ((m_1+m_2)n + (L-1)(m_1+m_3)n + (1+m_3)n)$, $((m_1+m_2)n + (L-1)(m_1+m_3)n + (1+m_3)n) \times (Lm_1+1)n$, respectively. For $2 \leq h \leq N-1$, B_h and for $1 \leq h \leq N-1$, E_h are square matrices of order $(m_1+m_2)n + (L-1)\ (m_1+m_3)n + (1+m_3)n$. F and B'_N are of orders $((m_1+m_2)n + (L-1)(m_1+m_3)n + (1+m_3)n) \times (m_2+Lm_3)n$ and $(m_2+Lm_3)n \times ((m_1+m_2)n + (L-1)(m_1+m_3)n + (1+m_3)n)$, respectively. For $h \geq N$, $A_0^{(h)}, A_1^{(h)}$ and for $h \geq N+1$, $A_2^{(h)}$ are square matrices of order $(m_2+Lm_3)n$. Define the entries $B_{0_{(i_1,j_1,k_1,l_1)}}^{(i_2,j_2,k_2,l_2)}$, $C_{0_{(i_1,j_1,k_1,l_1)}}^{(i_2,j_2,k_2,l_2)}$ and $B_{1_{(i_1,j_1,k_1,l_1)}}^{(i_2,j_2,k_2,l_2)}$ as transition submatrices which contain transitions of the form $(0, i_1, j_1, k_1, l_1) \to (0, i_2, j_2, k_2, l_2)$, $(0, i_1, j_1, k_1, l_1) \to (1, i_2, j_2, k_2, l_2)$ and $(1, i_1, j_1, k_1, l_1) \to (0, i_2, j_2, k_2, l_2)$, respectively. Define $E_{h_{(i_1,j_1,k_1,l_1)}}^{(i_2,j_2,k_2,l_2)}$, $B_{h_{(i_1,j_1,k_1,l_1)}}^{(i_2,j_2,k_2,l_2)}$, F and B'_N as transition submatrices which contain transitions of the form $(h,i_1,j_1,k_1,l_1)\rightarrow(h,i_2,j_2,k_2,l_2)$, where $1\le h\le N-1$, $(h,i_1,j_1,k_1,l_1)\rightarrow(h-1,i_2,j_2,k_2,l_2)$, where $2\le h\le N-1$, $(N-1,i_1,j_1,k_1,l_1)\rightarrow(N,i_2,j_2,k_2,l_2)$ and $(N,i_1,j_1,k_1,l_1)\rightarrow(N-1,i_2,j_2,k_2,l_2)$, respectively. Define the entries $A_2{}^{(h)(i_2,j_2,k_2,l_2)}_{\;(i_1,j_1,k_1,l_1)}$, $A_1{}^{(h)(i_2,j_2,k_2,l_2)}_{\;(i_1,j_1,k_1,l_1)}$ and $A_0{}^{(h)(i_2,j_2,k_2,l_2)}_{\;(i_1,j_1,k_1,l_1)}$ as transition submatrices which contain transitions of the form $(h,i_1,j_1,k_1,l_1)\rightarrow(h-1,i_2,j_2,k_2,l_2)$, where $h\ge N+1$, $(h,i_1,j_1,k_1,l_1)\rightarrow(h,i_2,j_2,k_2,l_2)$ and $(h,i_1,j_1,k_1,l_1)\rightarrow(h+1,i_2,j_2,k_2,l_2)$, where $h\ge N$, respectively. Since none or one event alone could take place in a short interval of time with positive probability, in general, a transition such as $(h_1,i_1,j_1,k_1,l_1)\rightarrow(h_2,i_2,j_2,k_2,l_2)$ has positive rate only for exactly one of h_1,i_1,j_1,k_1,l_1 different from h_2,i_2,j_2,k_2,l_2

$$B_{0\,(i_1,j_1,k_1,l_1)}^{(i_2,j_2,k_2,l_2)}=\begin{cases}T^o\alpha\otimes I_n & i_2=i_1+1,0\le i_1\le L-2;\,j_1=j_2=0;1\le k_1,k_2\le m_1;1\le l_1,l_2\le n\\ T^o\otimes I_n & i_1=L-1,i_2=L;\,j_1=j_2=0;1\le k_1,k_2\le m_1;1\le l_1,l_2\le n\\ T\oplus\Delta & i_1=i_2,0\le i_1\le L-1;\,j_1=j_2=0;1\le k_1,k_2\le m_1;1\le l_1,l_2\le n\\ \Delta & i_1=i_2=L;\,j_1=j_2=0;1\le l_1,l_2\le n\end{cases}$$

where

$$\Delta=D_0+(1-p)\begin{bmatrix}\delta_1 & & & \\ & \delta_2 & & \\ & & \ddots & \\ & & & \delta_n\end{bmatrix},$$

$$C_{0\,(i_1,j_1,k_1,l_1)}^{(i_2,j_2,k_2,l_2)}=\begin{cases}I_{m_1}\otimes pD_1, & 0\le i_1\le L-1;i_1=i_2;\,j_1=j_2=0;1\le k_1,k_2,\le m_1;1\le l_1,l_2\le n\\ pD_1, & i_2=i_1=L;\,j_1=j_2=0;1\le l_1,l_2\le n\end{cases}$$

$$B_{1\,(i_1,j_1,k_1,l_1)}^{(i_2,j_2,k_2,l_2)}=\begin{cases}S^0\alpha\otimes I_n, & i_1=i_2=0;\,j_1=1,\,j_2=0;1\le k_1\le m_2,1\le k_2\le m_1;1\le l_1,l_2\le n\\ U^0\alpha\otimes I_n, & 1\le i_1\le L;i_2=i_1-1;\,j_1=1,\,j_2=0;1\le k_1\le m_3,1\le k_2\le m_1;1\le l_1,l_2\le n.\end{cases}$$

For $1\le h\le N-1$,

$$E_{h\,(i_1,j_1,k_1,l_1)}^{(i_2,j_2,k_2,l_2)}=\begin{cases}T^0\alpha\otimes I_n, & 0\le i_1\le L-2,i_2=i_1+1;\,j_1=j_2=0;1\le k_1,k_2\le m_1;1\le l_1,l_2\le n\\ T^0\otimes I_n, & i_1=L-1,i_2=L;\,j_1=j_2=0;1\le k_1\le m_1;1\le l_1,l_2\le n\\ T\oplus\Delta-(h-1)\phi I_{m_1n}, & i_1=i_2,0\le i_1\le L-1;\,j_1=j_2=0;1\le k_1,k_2\le m_1;1\le l_1,l_2\le n\\ S\oplus\Delta-(h-1)\phi I_{m_1n}, & i_1=i_2=0,\,j_1=j_2=1,1\le k_1,k_2\le m_2,1\le l_1,l_2\le n\\ U\oplus\Delta-(h-1)\phi I_{m_1n}, & i_1=i_2,1\le i_1\le L;\,j_1=j_2=1,1\le k_1,k_2\le m_3,1\le l_1,l_2\le n\\ \Delta-(h-1)\phi I_n, & i_1=i_2=L;\,j_1=j_2=0;1\le l_1,l_2\le n.\end{cases}$$

For $2 \le h \le N-1$,

$$B_{h\,(i_1,j_1,k_1,l_1)}^{(i_2,j_2,k_2,l_2)} = \begin{cases} (h-1)\phi I_{m_1 n}, & 0 \le i_1 \le L-1, i_1 = i_2; j_1 = j_2 = 0; 1 \le k_1, k_2 \le m_1; 1 \le l_1, l_2 \le n \\ (h-1)\phi I_n, & i_1 = i_2 = L; j_1 = j_2 = 0; 1 \le k_1, k_2 \le m_1; 1 \le l_1, l_2 \le n \\ S^0 \beta \otimes I_n + (h-1)\phi I_{m_2 n}, & i_1 = i_2 = 0; j_1 = j_2 = 1; 1 \le k_1, k_2 \le m_2; 1 \le l_1, l_2 \le n \\ (h-1)\phi I_{m_3 n}, & 1 \le i_1 \le L, i_1 = i_2; j_1 = j_2 = 1; 1 \le k_1, k_2 \le m_3; 1 \le l_1, l_2 \le n \\ U^0 \beta \otimes I_n, & i_1 = 1, i_2 = 0; j_1 = j_2 = 1; 1 \le k_1 \le m_3, 1 \le k_2 \le m_2; 1 \le l_1, l_2 \le n \\ U^0 \gamma \otimes I_n, & 2 \le i_1 \le L, i_2 = i_1 - 1; j_1 = j_2 = 1; 1 \le k_1, k_2 \le m_3; 1 \le l_1, l_2 \le n \end{cases}$$

$$F_{(i_1,j_1,k_1,l_1)}^{(i_2,j_2,k_2,l_2)} = \begin{cases} e(m_1) \otimes (\beta \otimes pD_1), & i_1 = i_2 = 0; j_1 = 0, j_2 = 1; 1 \le k_1 \le m_1, 1 \le k_2 \le m_2; 1 \le l_1, l_2 \le n \\ I_{m_2} \otimes pD_1, & i_2 = i_1 = 0; j_1 = j_2 = 1; 1 \le k_1, k_2 \le m_2, 1 \le l_1, l_2 \le n \\ I_{m_3} \otimes pD_1, & i_2 = i_1, 1 \le i_1 \le L; j_1 = j_2 = 1; 1 \le k_1, k_2 \le m_2, 1 \le l_1, l_2 \le n \\ e(m_1) \otimes (\gamma \otimes pD_1), & 1 \le i_1 \le L-1; j_1 = 0, j_2 = 1; 1 \le k_1 \le m_1, 1 \le k_2 \le m_3; 1 \le l_1, l_2 \le n \\ \gamma \otimes pD_1, & i_1 = i_2 = L; j_1 = 0, j_2 = 1; 1 \le k_1 \le m_1, 1 \le k_2 \le m_3; 1 \le l_1, l_2 \le n \end{cases}$$

$$B'^{(i_2,j_2,k_2,l_2)}_{N\,(i_1,j_1,k_1,l_1)} = \begin{cases} S^0 \beta \otimes I_n + (N-1)\phi I_{m_2 n}, & i_1 = i_2 = 0; j_1 = j_2 = 1; 1 \le k_1, k_2 \le m_2; 1 \le l_1, l_2 \le n \\ U^0 \beta \otimes I_n, & i_1 = 1, i_2 = 0; j_1 = j_2 = 1; 1 \le k_1 \le m_3, 1 \le k_2 \le m_2; 1 \le l_1, l_2 \le n \\ U^0 \gamma \otimes I_n, & 2 \le i_1 \le L, i_2 = i_1 - 1; j_1 = j_2 = 1; 1 \le k_1, k_2 \le m_3; 1 \le l_1, l_2 \le n \\ (N-1)\phi I_{m_3 n}, & 1 \le i_1 \le L, i_2 = i_1; j_1 = j_2 = 1; 1 \le k_1, k_2 \le m_3; 1 \le l_1, l_2 \le n. \end{cases}$$

For $h \ge N+1$,

$$A_{2\,(i_1,j_1,k_1,l_1)}^{(h)(i_2,j_2,k_2,l_2)} = \begin{cases} S^0 \beta \otimes I_n + (h-1)\phi I_{m_2 n}, & i_1 = i_2 = 0; j_1 = j_2 = 1; 1 \le k_1, k_2 \le m_2; 1 \le l_1, l_2 \le n \\ U^0 \beta \otimes I_n, & i_1 = 1, i_2 = 0; j_1 = j_2 = 1; 1 \le k_1 \le m_3, 1 \le k_2 \le m_2; 1 \le l_1, l_2 \le n \\ U^0 \gamma \otimes I_n, & i_2 = i_1 - 1, 2 \le i_2 \le L; j_1 = j_2 = 1; 1 \le k_1, k_2 \le m_3; 1 \le l_1, l_2 \le n \\ (h-1)\phi I_{m_3 n}, & i_2 = i_1, 1 \le i_2 \le L; j_1 = j_2 = 1; 1 \le k_1, k_2 \le m_3; 1 \le l_1, l_2 \le n. \end{cases}$$

For $h \ge N$,

$$A_{1\,(i_1,j_1,k_1,l_1)}^{(h)(i_2,j_2,k_2,l_2)} = \begin{cases} S \oplus \Delta - (h-1)\phi I_{m_2 n}, & i_1 = i_2 = 0, j_1 = j_2 = 1, 1 \le k_1, k_2 \le m_2, 1 \le l_1, l_2 \le n \\ U \oplus \Delta - (h-1)\phi I_{m_3 n}, & i_1 = i_2, 1 \le i_1 \le L; j_1 = j_2 = 1, 1 \le k_1, k_2 \le m_3, 1 \le l_1, l_2 \le n. \end{cases}$$

$$A_{0\,(i_1,j_1,k_1,l_1)}^{(h)(i_2,j_2,k_2,l_2)} = \begin{cases} I_{m_2} \otimes pD_1, & i_1 = i_2 = 0; j_1 = j_2 = 1; 1 \le k_1, k_2 \le m_2; 1 \le l_1, l_2 \le n \\ I_{m_3} \otimes pD_1, & i_1 = i_2, 1 \le i_1 \le L; j_1 = j_2 = 1; 1 \le k_1, k_2 \le m_3; 1 \le l_1, l_2 \le n. \end{cases}$$

Remarks: When $L = 0$ (that is, no item processed during vacation) the problem discussed reduces to classical N -policy.

3. Steady State Analysis

3.1. Stability condition

Lemma 3.1. *The system under consideration is stable.*

Proof. We use the following result to prove this(see Tweedie [15]).

Result (Tweedie [15]**)** Let $\{X(t)\}$ be a Markov process with discrete state space $\mathcal{S}$ and rates of transition q_{sr}, $s, r \in \mathcal{S}$, $\sum_r q_{sr} = 0$. Assume that there exist
1. a function $\psi(s)$, $s \in \mathcal{S}$, which is bounded from below (this function is said to be a Lyapunov or test function);
2. a positive number ϵ such that:

- variables $y_s = \sum_{r \neq s} q_{sr}(\psi(r) - \psi(s)) < \infty$ for all $s \in S$;
- $y_s \leq -\epsilon$ for all $s \in S$ except perhaps a finite number of states.

Then the process $\{X(t)\}$ is regular and ergodic.

For the model under discussion, we consider the following test function:

$$\psi(s) = \psi(h, i, j, k, l) = h .$$

The mean drifts

$$\begin{aligned} y_s &= \sum_{r \neq s} q_{sr} \left(\psi(r) - \psi(s) \right) \\ &= q_{s,s+1} - q_{s,s-1}. \end{aligned} \tag{1}$$

We have $q_{s,s+1} = r_1$, say (a constant) and $q_{s,s-1} = r_2 + (s-1)\phi$, where r_2 is a constant. Hence from (1), $y_s = r_1 - r_2 - (s-1)\phi$, which depends only on the level s. Now,

$$lim_{s \to \infty} y_s = -\infty.$$

Thus the assumptions of Tweedie's result hold and hence the Markov process under consideration is regular and ergodic. Hence the system is stable.

3.2. Steady state vector

By finite truncation method we get steady state vectors of the LDQBD approximately. In this method, we truncate the infinitesimal genarator at a finite level K. The level K is chosen in such a way that probability of customer loss due to truncation is small. To get an appropriate level,say , K_f, we start with an initial value for K and increasing it in unit steps until a properly chosen cut-off criterion is satisfied. Here, we use the algorithm by Artalejo *et al.* [1], the steps of which are explained below.

With K as cut-off level, the modified generator is

$$\bar{Q}(K)=\begin{bmatrix} B_0 & C_0 & & & & & & & & \\ B_1 & E_1 & I\otimes pD_1 & & & & & & & \\ & B_2 & E_2 & I\otimes pD_1 & & & & & & \\ & & \ddots & \ddots & \ddots & & & & & \\ & & & B_{N-2} & E_{N-2} & I\otimes pD_1 & & & & \\ & & & & B_{N-1} & E_{N-1} & F & & & \\ & & & & & B'_N & A_1^{(N)} & A_0^{(N)} & & \\ & & & & & & A_2^{(N+1)} & A_1^{(N+1)} & A_0^{(N+1)} & \\ & & & & & & \ddots & \ddots & \ddots & \\ & & & & & & & A_2^{(K-1)} & A_1^{(K-1)} & A_0^{(K-1)} \\ & & & & & & & & A_2^{(K)} & \theta^{(K)} \end{bmatrix},$$

where $\theta^{(K)}=A_1^{(K)}+A_0^{(K)}$. Let $\bar{\pi}$ be the stationary distribution of $\bar{Q}(K)$ which satisfies

$$\begin{aligned} \bar{\boldsymbol{\pi}}\bar{Q}(K)&=0,\\ \bar{\boldsymbol{\pi}}\boldsymbol{e}&=1, \end{aligned} \tag{2}$$

where $\bar{\boldsymbol{\pi}}=[\bar{\boldsymbol{\pi}}(0),\bar{\boldsymbol{\pi}}(1),\ldots,\bar{\boldsymbol{\pi}}(K)]$. Define $\boldsymbol{y}=[\boldsymbol{y}_0(K),\boldsymbol{y}_1(K)]$ with

$$\begin{aligned} \boldsymbol{y}_0(K)&=[\bar{\boldsymbol{\pi}}(0),\bar{\boldsymbol{\pi}}(1),\ldots,\bar{\boldsymbol{\pi}}(K-1)],\\ \boldsymbol{y}_1(K)&=\bar{\boldsymbol{\pi}}(K). \end{aligned}$$

Now $\boldsymbol{y}(K,i)=\bar{\boldsymbol{\pi}}(i),\ 0\le i\le K$. Here $\boldsymbol{y}_0(K)$ is a row vector of dimension $m=(Lm_1+1)n+(N-1)[m_2n+L(m_1+m_3)n+n]+(K-N)(m_2+Lm_3)n$ and $\boldsymbol{y}_1(K)$ is a row vector of dimension $(m_2+Lm_3)n$. Now from (2), we have

$$[\boldsymbol{y}_0(K),\boldsymbol{y}_1(K)]\begin{bmatrix} H_{00}(K) & H_{01}(K)\\ H_{10}(K) & H_{11}(K)\end{bmatrix}=[\mathbf{0}_m,\mathbf{0}_{(m_2+Lm_3)n}], \tag{3}$$

where $H_{00}(K)$ is obtained from $\bar{Q}(K)$ by deleting the last column matrices and last row matrices. $H_{01}(K)=[0,0,\ldots,0,A_0^{(K-1)}]^T$, $H_{10}(K)=[0,0,\ldots,0,A_2^{(K)}]$ and $H_{11}(K)=\theta^{(K)}$. These are block structured matrices with $K\times K, K\times 1, 1\times K$ and 1×1 blocks respectively. $\mathbf{0}_m$ and $\mathbf{0}_{(m_2+Lm_3)n}$ are row vectors of dimensions m and $(m_2+Lm_3)n$, respectively, with all entries equal to zero. From (3), we get

$$\boldsymbol{y}_1(K)H_{10}(K)H_{00}^{-1}(K)=-\boldsymbol{y}_0(K), \tag{4}$$

$$\boldsymbol{y}_1(K)[H_{11}(K)-H_{10}(K)H_{00}^{-1}(K)H_{01}(K)]=\mathbf{0}_{(m_2+Lm_3)n}. \qquad (5)$$

Also we have

$$H_{00}(K)=\begin{bmatrix} H_{00}(K-1) & H_{01}(K-1) \\ J_0(K-1) & J_1(K-1) \end{bmatrix},$$

where

$$J_0(K-1)=[0,\ldots,0,A_2^{(K-1)}],$$
$$J_1(K-1)=A_1^{(K-1)}.$$

The inverse of matrix $H_{00}(K)$ can be determined using theorem 4.2.4 in Hunter [9] as

$$H_{00}^{-1}(K)=\begin{bmatrix} M_{00}(K) & M_{01}(K) \\ M_{10}(K) & M_{11}(K) \end{bmatrix},$$

where

$$M_{00}(K)=[H_{00}(K-1)-H_{01}(K-1)J_1^{-1}(K-1)J_0(K-1)]^{-1},$$
$$M_{01}(K)=-J_1^{-1}(K-1)J_0(K-1)M_{00}(K),$$
$$M_{11}(K)=[J_1(K-1)-J_0(K-1)H_{00}^{-1}(K-1)H_{01}(K-1)]^{-1},$$
$$M_{01}(K)=-H_{00}^{-1}(K-1)H_{01}(K-1)M_{11}(K).$$

Now we can see that the structure of the block matrices $H_{01}(K-1)$ and $J_0(K-1)$ simplify the above set of equations. We have

$$H_{00}^{-1}(K-1)H_{01}(K-1)=\begin{bmatrix} M_{01}(K-1) \\ M_{11}(K-1) \end{bmatrix}A_0^{(K-2)}.$$

Also $J_0(K-1)H_{00}^{-1}(K-1)H_{01}(K-1)=A_2^{(K-1)}M_{11}(K-1)A_0^{(K-2)}$.

By example 4.2.2 (Hunter [9]), we have $(X+AYB)^{-1}=X^{-1}-X^{-1}A(Y^{-1}+BX^{-1}A)^{-1}BX^{-1}$.
Then we have

$$(X+AYB)^{-1}=[I-X^{-1}A(Y^{-1}+BX^{-1}A)^{-1}B]X^{-1}.$$

Here, we have $X=H_{00}(K-1), A=-H_{01}(K-1), Y=J_1^{-1}(K-1)$ and $B=J_0(K-1)$. Finally, we get

$$M_{00}(K)=[I-M_{01}(K)J_0(K-1)]H_{00}^{-1}(K-1),$$
$$M_{11}(K)=[J_1(K-1)-A_2^{(K-1)}M_{11}(K-1)A_0^{(K-2)}]^{-1},$$
$$M_{01}(K)=-\begin{bmatrix} M_{01}(K-1) \\ M_{11}(K-1) \end{bmatrix}A_0^{(K-2)}M_{11}(K),$$
$$M_{10}(K)=-J_1^{-1}(K-1)J_0(K-1)M_{00}(K).$$

Thus the computation of the vector $\boldsymbol{y}_1(K)$ reduces to solving the system of equations (5) subject to the normalizing condition

$$\overline{\boldsymbol{\pi}}(K)[\boldsymbol{e} - H_{10}(K)H_{00}^{-1}(K)\boldsymbol{e}] = 1 .$$

The vector $\boldsymbol{y}_0(K)$ can be solved by substituting $\boldsymbol{y}_1(K)$ in (4). To get the cut-off value, successive increments of K are made, starting from $N+2$ and we stop at the point $K = K_c$ when

$$max_{0\le i\le K_c} \| \boldsymbol{y}(K_c,i) - \boldsymbol{y}(K_c - 1,i)\|_\infty < \epsilon ,$$

where $\epsilon > 0$ is infinitesimal quantity and $\|.\|_\infty$ is the infinity norm (see Goswami and Selavaraju [7]).

Now we suppose that $x_{h,i,j,k,l}$ represent the steady state probability that system is in state (h,i,j,k,l).

3.3. Distribution of time until the number of customers hit N

We show that this is a phase type distribution where the underlying Markov process has state space

$$\{(h,i,j,k): 0 \le h \le N-1, 0 \le i \le L-1, 1 \le j \le m_1, 1 \le k \le n\}$$
$$\bigcup\{(h,L,k): 0 \le h \le N-1, 1 \le k \le n\} \bigcup \{*\},$$

where $*$ denotes the absorbing state indicating the number of customers reaching N. The infinitesimal generator is

$$\mathbf{V_1} = \begin{bmatrix} V_1 & V_1^{(0)} \\ \mathbf{0} & 0 \end{bmatrix}, \quad \text{where}$$

$$\mathbf{V_1} = \begin{bmatrix} H_0 & I_{Lm_1+1}\otimes pD_1 & & & & \\ & H_0 & I_{Lm_1+1}\otimes pD_1 & & & \\ & G_2 & H_2 & I_{Lm_1+1}\otimes pD_1 & & \\ & \ddots & \ddots & \ddots & & \\ & & G_{N-2} & H_{N-2} & I_{Lm_1+1}\otimes pD_1 \\ & & & G_{N-1} & H_{N-1} \end{bmatrix},$$

$$V_1^0 = \begin{bmatrix} \mathbf{0} \\ \vdots \\ \mathbf{0} \\ \boldsymbol{e}(Lm_1+1)\otimes p\delta \end{bmatrix}, \quad \text{with} \quad H_0 = \begin{bmatrix} T\oplus\Delta & T^0\alpha\otimes I_n & 0 & 0 \\ 0 & T\oplus\Delta & T^0\alpha\otimes I_n & 0 \\ 0 & 0 & T\oplus\Delta & T^0\otimes I_n \\ 0 & 0 & 0 & \Delta \end{bmatrix}.$$

For $2 \le h \le N-1$,

$$G_h = (h-1)\phi I_{(Lm_1+1)n}$$

and

$$H_h = \begin{bmatrix} T \oplus \Delta - (h-1)\phi I_{m_1 n} & T^0\alpha \otimes I_n & 0 & 0 \\ 0 & T \oplus \Delta - (h-1)\phi I_{m_1 n} & T^0\alpha \otimes I_n & 0 \\ 0 & 0 & T \oplus \Delta - (h-1)\phi I_{m_1 n} & T^0 \otimes I_n \\ 0 & 0 & 0 & \Delta - (h-1)\phi I_{m_1 n} \end{bmatrix}.$$

The initial probability vector is

$$\psi_1 = (\frac{1}{d_1})(w_{0,0,1,1}, \cdots, w_{0,0,1,n}, \cdots, w_{0,0,m_1,1}, \cdots, w_{0,0,m_1,n} \cdots, w_{0,L-1,m_1,1}, \cdots, w_{0,L-1,m_1,n}, 0),$$

where

$$w_{0,0,k,l} = \sum_{k'=1}^{m_2} \frac{\sigma_{k'}\alpha_k}{-d_{ll}^{(0)} - S_{k'k'}} x_{1,0,1,k',l} + \sum_{k'=1}^{m_3} \frac{\tau_{k'}\alpha_k}{-d_{ll}^{(0)} - U_{k'k'}} x_{1,1,1,k',l},$$

$$w_{0,i,k,l} = \sum_{k'=1}^{m_3} \frac{\tau_{k'}\alpha_k}{-d_{ll}^{(0)} - U_{k'k'}} x_{1,i+1,1,k',l}, \quad \text{with } 1 \le i \le L-1,$$

and

$d_1 = \sum_{l=1}^{n}\sum_{i=0}^{L-1}\sum_{k=1}^{m_1} w_{0,i,k,l}$, where $\mathbf{0}$ is a zero matrix of order $1 \times ((N-1)Lm_1 n + n)$.

Here, $\sigma_{k'}$ represents the absorption rate to phase k' from $PH(\beta, S)$, $\tau_{k'}$ represents the absorption rate to phase k' from $PH(\gamma, U)$, $S_{k'k'}$ represent the $k'k'$ th entry of S, $U_{k'k'}$ represent the $k'k'$ th entry of U, α_k denote the kth component of the intial distribution of inventory processing and $d_{ll}^{(0)}$ represent the diagonal entry in l th row of D_0.

3.4. Some other Performance Measures

- Probability that the server is idle,

$$P_{idle} = \sum_{h=0}^{N-1}\sum_{l=1}^{n} x_{h,L,0,l}.$$

- The fraction of time the server is idle waiting for Nth customer,

$$P_{id} = \sum_{l=1}^{n} x_{N-1,L,0,l}.$$

• The fraction of time the server is busy in inventory production,

$$P_{pro} = \sum_{h=0}^{N-1}\sum_{i=0}^{L-1}\sum_{k=1}^{m_1}\sum_{l=1}^{n} x_{h,i,0,k,l}.$$

• The fraction of time the server is busy serving,

$$P_{ser} = \sum_{h=1}^{K}\sum_{k=1}^{m_2}\sum_{l=1}^{n} x_{h,0,1,k,l} + \sum_{h=1}^{K}\sum_{i=1}^{L}\sum_{k=1}^{m_3}\sum_{l=1}^{n} x_{h,i,1,k,l}.$$

• Expected number of customers in the system,

$$\begin{aligned} E(S) = &\sum_{h=1}^{N-1}\sum_{i=0}^{L-1}\sum_{k=1}^{m_1}\sum_{l=1}^{n} hx_{h,i,0,k,l} + \sum_{h=1}^{N-1}\sum_{l=1}^{n} hx_{h,L,0,l} + \sum_{h=1}^{K}\sum_{k=1}^{m_2}\sum_{l=1}^{n} hx_{h,0,1,k,l} \\ &+ \sum_{h=1}^{K}\sum_{i=1}^{L}\sum_{k=1}^{m_3}\sum_{l=1}^{n} hx_{h,i,1,k,l}. \end{aligned} \tag{6}$$

• Expected number of items in the inventory,

$$E(it) = \sum_{h=0}^{N-1}\sum_{i=1}^{L-1}\sum_{k=1}^{m_1}\sum_{l=1}^{n} ix_{h,i,0,k,l} + \sum_{h=0}^{N-1}\sum_{l=1}^{n} Lx_{h,L,0,l} + \sum_{h=1}^{K}\sum_{i=1}^{L}\sum_{k=1}^{m_3}\sum_{l=1}^{n} ix_{h,i,1,k,l}. \tag{7}$$

• Expected rate at which the inventory processing is switched on,

$$E(ipo) = \sum_{k=1}^{m_2}\sum_{l=1}^{n} \sigma_k x_{1,0,1,k,l} + \sum_{i=1}^{L}\sum_{k=1}^{m_3}\sum_{l=1}^{n} \tau_k x_{1,i,1,k,l}.$$

4. Special Cases

1. $p = 1, \phi = 0$

   In this case, the present model reduces to Divya *et al.* [4]. We see that the model can be studied as a LIQBD process.

2. $\phi = 0$

   In this case also, the model can be studied as a LIQBD process with obvious modifications in Divya *et al.* [4].

From now on we concentrate in the case $\phi = 0$.

First, we find the LST of the waiting time distribution.

4.1. Waiting time analysis

To find the waiting time of a customer who joins for service at time t, we have to consider different possibilities depending on the status of server at that time.The server may be on vacation or in normal mode. Let $W(t)$ be the waiting time of a customer in the system who arrives at time t and $W^*(s)$ be the corresponding LST.

Case I. (Vacation mode)

Let Ev_1 denote the event that the tagged customer immedietly after his arrival finds the system in the state $(h'+1, i', 0, k', l')$ or in the state $(h'+1, L, 0, l')$, where $0 \le h' \le N-2$, $0 \le i' \le L-1$, $1 \le k' \le m_1$, $1 \le l' \le n$.

In this case, the waiting time is the time until absorption in a Markov process whose state space is given by $\{(h,i,k_1,l): 1 \le h \le N-1, 0 \le i \le L-1, 1 \le k_1 \le m_1, 1 \le l \le n\} \cup \{(h,L,l): 1 \le h \le N-1, 1 \le l \le n\} \cup \{(h^*,0,k_2): 1 \le h^* \le N-1, 1 \le k_2 \le m_2\} \cup \{(h^*,i,k_3): 1 \le h^* \le N-1, 1 \le i \le L, 1 \le k_3 \le m_3\} \cup \{*\}$ where (h,i,k_1,l) denote the states that correspond to the server being on vacation with h customers in the system , i, items in inventory, k_1, the processing phase and l, the arrival phase, (h,L,l) denote the state that correspond to the server being on vacation mode with h customers in the system, L items in inventory and l, the arrival phase. $(h^*,0,k_2)$ denote the states that correspond to the tagged customer being in the position h^* when the server is on normal mode, k_2, the service phase when there is no processed item, (h^*,i,k_3) denote the states that correspond to the tagged customer being in position h^* when the server is in normal mode with i processed items in the inventory and k_3 denote the service phase and $*$ denote the absorbing state indicating the service completion of the tagged customer. Thus the conditional waiting time can be studied by a phase type distribution with representation $PH(\psi_1, W_1)$ where

$$W_1 = \begin{bmatrix} M_{11} & M_{12} \\ 0 & M_{22} \end{bmatrix}, \quad W_1^0 = \begin{bmatrix} \mathbf{0} \\ M^0 \end{bmatrix},$$

where

$$M^0 = \begin{bmatrix} M^{00} \\ \mathbf{0} \end{bmatrix}, \quad \text{with} \quad M^{00} = \begin{bmatrix} S^0 \\ e(L) \otimes U^0 \end{bmatrix},$$

$$M_{11} = \begin{bmatrix} E & I_{Lm_1+1} \otimes D_1 & & \\ & \ddots & \ddots & \\ & & E & I_{Lm_1+1} \otimes D_1 \\ & & & E \end{bmatrix}, \quad \text{where}$$

$$E = \begin{bmatrix} T \oplus \Delta & T^0\alpha \otimes I_n & & & \\ & \ddots & \ddots & & \\ & & T \oplus \Delta & T^0\alpha \otimes I_n & \\ & & & T \oplus \Delta & T^0 \otimes I_n \\ & & & & \Delta \end{bmatrix},$$

$M_{12} = e_{N-1}(N-1)e_{h_1}'(N-1) \otimes F$, where

$$F = \begin{bmatrix} e(m_1) \otimes (p\delta \otimes \beta) & & & \\ & e(m_1) \otimes (p\delta \otimes \gamma) & & \\ & & \ddots & \\ & & & p\delta \otimes \gamma \end{bmatrix},$$

$$M_{22} = \begin{bmatrix} G & & & \\ H & G & & \\ & \ddots & \ddots & \\ & & H & G \end{bmatrix}, \quad \text{where } G = \begin{bmatrix} S & \\ & I_L \otimes U \end{bmatrix}, \quad H = \begin{bmatrix} S^0\beta & 0 & 0 \\ U^0\beta & 0 & 0 \\ 0 & I_{L-1} \otimes (U^0\gamma) & 0 \end{bmatrix}.$$

Thus the conditional LST,

$$W^*(s \mid Ev_1) = \psi_1(sI - W_1)^{-1}W_1^0,$$

where ψ_1 is the initial probabilty vector which ensures that the Markov chain always starts from the level h.

Case II. (Normal mode)

Let Ev_2 denote the event that the tagged customer immedietly after his joining finds the system in the state $(h'+1, 0, 1, k'', l')$, where $h' \geq 1$, $1 \leq k'' \leq m_2$, $1 \leq l' \leq n$ or in the state $(h'+1, i', 1, k''', l')$, where $1 \leq h' \leq N-1$, $1 \leq i' \leq L-N+h'$, $1 \leq k''' \leq m_3$, $1 \leq l' \leq n$ or in the state $(h'+1, i', 1, k''', l')$, where $h' \geq N$, $1 \leq i' \leq L$, $1 \leq k''' \leq m_3$, $1 \leq l' \leq n$.

In this case, the waiting time is the time until absorption in a Markov process whose state space is given by $\{(h,0,k): 2 \leq h \leq K, 1 \leq k \leq m_2\} \bigcup \{(h,i,k): 2 \leq h \leq N-1, 1 \leq i \leq L-N+h, 1 \leq k \leq m_3\} \bigcup \{(h,i,k): N \leq h \leq K, 1 \leq i \leq L, 1 \leq k \leq m_3\} \bigcup \{*\}$ where $(h,0,k)$ denote the states that correspond to the server being in normal mode with h customers in the system, service phase k when there is no processed item, (h,i,k) denote the states that correspond to the server being in normal mode with h customers in the system, service phase k when there are i processed items and * denote the absorbing state indicating the service completion of the tagged customer and K is chosen in such a way that $P\left(\sum_{h=0}^{K} x_h \boldsymbol{e} > 1-\epsilon\right) \to 0$ for every $\epsilon > 0$. Thus the conditional waiting time can be studied by a truncated phase type distribution with representation $Ph(\psi_2, W_2)$ where

$$W_2 = \begin{bmatrix} G_1 & & & & & & & \\ H_1 & G_2 & & & & & & \\ & \ddots & \ddots & & & & & \\ & & H_{N-2} & G_{N-1} & & & & \\ & & & H_{N-1} & G & & & \\ & & & & H & G & & \\ & & & & & \ddots & \ddots & \\ & & & & & & H & G \end{bmatrix},$$

$$W_2^{\ 0} = \begin{bmatrix} E^0 \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix}, \quad \text{where} \quad E^0 = \begin{bmatrix} S^0 \\ e(L-N+1) \otimes U^0 \end{bmatrix}.$$

For $1 \le h \le N-1$,

$$G_h = \begin{bmatrix} S & \\ & I_{L-N+h} \otimes U \end{bmatrix}, \quad H_h = \begin{bmatrix} S^0\beta & 0 \\ U^0\beta & 0 \\ 0 & I_{L-N+h} \otimes U^0\gamma \end{bmatrix},$$

$$E = \begin{bmatrix} S & \\ & I_L \otimes U \end{bmatrix}, \quad F = \begin{bmatrix} S^0\beta & 0 & 0 \\ U^0\beta & 0 & 0 \\ 0 & I_{L-1} \otimes U^0\gamma & 0 \end{bmatrix}.$$

Thus the conditional LST,

$$W^*(s \mid Ev_2) = \psi_2 (sI - W_2)^{-1} W_2^{\ 0},$$

where ψ_2 is the initial probabilty vector which ensures that the Markov chain always starts from the level h.

Let $w_{h,i,j,k,l}$ and $w_{h,L,0,l}$ denote the probabaility that the tagged customer finds the system in the state (h,i,j,k,l) and $(h,L,0,l)$ respectively immedietly after his arrival. Then

$$w_{h,i,0,k_1,l} = \sum_{l'=1}^{n} \frac{p d_{l'l}^{(1)}}{-d_{l'l'}^{(0)} - (1-p)\delta_{l'} - T_{k_1k_1}} x_{h-1,i,0,k_1,l'},$$
$$1 \le h \le N-1,\ 0 \le i \le L-1, 1 \le k_1 \le m_1,\ 1 \le l \le n$$

$$w_{h,L,0,l} = \sum_{l'=1}^{n} \frac{p d_{l'l}^{(1)}}{-d_{l'l'}^{(0)} - (1-p)\delta_{l'}} x_{h-1,L,0,l'}, \qquad 1 \le h \le N-1, 1 \le l \le n$$

$$w_{h,0,1,k_2,l} = \sum_{l'=1}^{n} \frac{pd_{l'l}^{(1)}}{-d_{l'l'}^{(0)} - (1-p)\delta_{l'} - S_{k_2k_2}} x_{h-1,0,1,k_2,l'}, \quad 2 \le h \le N-1, \text{ or } h \ge N+1,$$

$$1 \le k_2 \le m_2, 1 \le l \le n$$

$$w_{N,0,1,k_2,l} = \sum_{l'=1}^{n}\sum_{k_1=1}^{m_1} \frac{pd_{l'l}^{(1)}\beta_{k_2}}{-d_{l'l'}^{(0)} - (1-p)\delta_{l'} - T_{k_1k_1}} x_{N-1,0,0,k_1,l'}$$

$$+\sum_{l'=1}^{n} \frac{pd_{l'l}^{(1)}}{-d_{l'l'}^{(0)} - (1-p)\delta_{l'} - S_{k_2k_2}} x_{N-1,0,1,k_2,l'}, \quad 1 \le k_2 \le m_2, 1 \le l \le n$$

$$w_{h,i,1,k_3,l} = \sum_{l'=1}^{n} \frac{pd_{l'l}^{(1)}}{-d_{l'l'}^{(0)} - (1-p)\delta_{l'} - U_{k_3k_3}} x_{h-1,i,1,k_3,l'},$$

$$2 \le h \le N-1, 1 \le L-N+h-1, 1 \le k_3 \le m_3, 1 \le l \le n$$

$$w_{h,i,1,k_3,l} = \sum_{l'=1}^{n} \frac{pd_{l'l}^{(1)}}{-d_{l'l'}^{(0)} - (1-p)\delta_{l'} - U_{k_3k_3}} x_{h-1,i,1,k_3,l'}, \ h \ge N+1, 1 \le i \le L, 1 \le k_3 \le m_3, 1 \le l \le n$$

$$w_{N,i,1,k_3,l} = \sum_{l'=1}^{n}\sum_{k_1=1}^{m_1} \frac{pd_{l'l}^{(1)}\gamma_{k_3}}{-d_{l'l'}^{(0)} - (1-p)\delta_{l'} - T_{k_1k_1}} x_{N-1,i,0,k_1,l'}$$

$$+\sum_{l'=1}^{n} \frac{pd_{l'l}^{(1)}}{-d_{l'l'}^{(0)} - (1-p)\delta_{l'} - S_{k_2k_2}} x_{N-1,i,1,k_2,l'}, \quad 1 \le i \le L, 1 \le k_2 \le m_2, 1 \le l \le n.$$

Thus we have the following theorem.

Theorem 4.1. *The LST of the waiting time is given by*

$$W^*(s) = \frac{1}{d_2}[\sum_{h'=1}^{N-1}\sum_{i'=0}^{L-1}\sum_{k'=1}^{m_1}\sum_{l'=1}^{n}\psi_1(sI-W_1)^{-1}W_1^0 w_{h',i',0,k',l'} + \sum_{h'=1}^{N-1}\sum_{l'=1}^{n}\psi_1(sI-W_1)^{-1}W_1^0 w_{h',L,0,l'}$$
$$+\sum_{h'=1}^{\infty}\sum_{k''=1}^{m_2}\sum_{l'=1}^{n}\psi_2(sI-W_2)^{-1}W_2^0 w_{h',0,1,k'',l'} + \sum_{h'=1}^{N}\sum_{i'=1}^{L-N+h'-1}\sum_{k'''=1}^{m_3}\sum_{l'=1}^{n}\psi_2(sI-W_2)^{-1}W_2^0 w_{h',i',1,k''',l'}$$
$$+\sum_{h'=N+1}^{\infty}\sum_{i'=1}^{L}\sum_{k'''=1}^{m_3}\sum_{l'=1}^{n}\psi_2(sI-W_2)^{-1}W_2^0 w_{h',i',1,k''',l'}], \tag{8}$$

where

$$d_2 = \sum_{h'=1}^{N-1}\sum_{i'=0}^{L-1}\sum_{k'=1}^{m_1}\sum_{l'=1}^{n} w_{h',i',0,k',l'} + \sum_{h'=1}^{N-1}\sum_{l'=1}^{n} w_{h',L,0,l'} + \sum_{h'=1}^{\infty}\sum_{k''=1}^{m_2}\sum_{l'=1}^{n} w_{h',0,1,k'',l'}$$
$$+\sum_{h'=1}^{N}\sum_{i'=1}^{L-N+h'-1}\sum_{k'''=1}^{m_3}\sum_{l'=1}^{n} w_{h',i',1,k''',l'} + \sum_{h'=N+1}^{\infty}\sum_{i'=1}^{L}\sum_{k'''=1}^{m_3}\sum_{l'=1}^{n} w_{h',i',1,k''',l'}. \tag{9}$$

Now,we assume that each customer receives a reward of R units after service completion and he has to pay a price q $(0 \le q < R)$ for an item. Let h_w denote the waiting cost per unit time of a customer in the system.

4.2. Individual equillibrium strategy

Define

$$F_1(p) = R - q - h_w E(W).$$

We have to find an equillibrium strategy according to which the customers join the system.

4.3. Revenue maximization

We have to find an optimal price q to maximize the revenue of the server given by

$$F_2(q) = p_e q \pi^* D_1 e - h_1 E(S) - h_2 E(it) - cE(ipo),$$

where

h_1 : holding cost/unit customer/unit time,

h_2 : holding cost/unit item/unit time,

c : switching on cost of inventory processing/unit time,

p_e : Individual equillibrium strategy corresponding to q.

4.4. Social optimal strategy

Next we consider social optimal strategy. For a given price q and a joining probability p, the surplus of all customers S_1 and the server revenue S_2 are given by

$$S_1 = \pi^* p D_1 e(R - q - h_w E(W)) \quad \text{and} \quad S_2 = pq\pi^* D_1 e - h_1 E(S) - h_2 E(it) - cE(ipo).$$

Therefore, the expected social welfare per unit time is,

$$F_3(p) = S_1 + S_2 = \pi^* p D_1 e(R - h_w E(W)) - h_1 E(S) - h_2 E(it) - cE(ipo).$$

4.5. Numerical results

We fix $N = 3$, $L = 2$, $\alpha = \beta = \begin{bmatrix} 1 & 0 \end{bmatrix}$, $\gamma = \begin{bmatrix} 0.8 & 0.2 \end{bmatrix}$, $T = \begin{bmatrix} -50 & 50 \\ 0 & -50 \end{bmatrix}$,

$S = \begin{bmatrix} -80 & 80 \\ 0 & -80 \end{bmatrix}$, $U = \begin{bmatrix} -150 & 150 \\ 0 & -150 \end{bmatrix}$, $R = 75$, $q = 60$, $h_w = 50$, $h_1 = 2$, $h_2 = 1$, $c = 30$.

We find the individual optimum and social optimum corresponding to the above parameters.

Table 1. Effect of p on various performance measures, when $D_0 = (-20),\ D_1 = (20)$.

p	$E(W)$	$E(S)$	$E(it)$	$E(ipo)$	F_1	F_3
0.1	0.5045	1.0248	1.8794	0.6445	-10.2243	76.2876
0.2	0.2571	1.0514	1.7601	1.2429	2.1430	238.8498
0.3	0.1762	1.0806	1.6421	1.7921	6.1878	339.5609
0.4	0.1368	1.1129	1.5254	2.2882	8.1599	472.8811
0.5	0.1138	1.1486	1.4102	2.7271	9.3077	607.5573
0.6	0.0991	1.1881	1.2965	3.1041	10.0449	743.7425
0.7	0.0891	1.2316	1.1848	3.4149	10.5461	881.5515
0.8	0.0821	1.2794	1.0752	3.6551	10.8954	1021.0406
0.9	0.0773	1.3325	0.9680	3.8207	11.1356	1162.1853
1.0	0.0743	1.3925	0.8635	3.9081	11.2870	1304.8471

Table 2. Effect of p on various performance measures, when $D_0 = (-25),\ D_1 = (25)$.

p	$E(W)$	$E(S)$	$E(it)$	$E(ipo)$	F_1	F_3
0.1	0.4052	1.0312	1.8494	0.7985	-5.2584	108.9862
0.2	0.2084	1.0657	1.7009	1.5239	4.5807	273.3550
0.3	0.1446	1.1045	1.5545	2.1694	7.7701	439.4306
0.4	0.1138	1.1486	1.4102	2.7271	9.3077	607.5573
0.5	0.0962	1.1986	1.2684	3.1882	10.1881	778.0391
0.6	0.0853	1.2549	1.1297	3.5441	10.7364	951.0852
0.7	0.0783	1.3187	0.9946	3.7865	11.0843	1126.7485
0.8	0.0743	1.3925	0.8635	3.9081	11.2869	1304.8471
0.9	0.0728	1.4827	0.7371	3.9027	11.3611	1484.8406
1.0	0.0741	1.6035	0.6156	3.7651	11.2950	1665.6008

Table 3. Effect of p on various performance measures, when $D_0 = (-30),\ D_1 = (30)$.

p	$E(W)$	$E(S)$	$E(it)$	$E(ipo)$	F_1	F_3
0.1	0.3392	1.0378	1.8196	0.9496	-1.9586	141.7398
0.2	0.1762	1.0806	1.6421	1.7921	6.1878	339.5609
0.3	0.1240	1.1303	1.4676	2.5151	8.8023	540.0395
0.4	0.0991	1.1881	1.2965	3.1041	10.0449	743.7425
0.5	0.0853	1.2549	1.1297	3.5441	10.7365	951.0857
0.6	0.0773	1.3324	0.9680	3.8207	11.1356	1162.1853
0.7	0.0734	1.4260	0.8124	3.9215	11.3322	1376.6660
0.8	0.0732	1.5501	0.6636	3.8363	11.3401	1593.3114
0.9	0.0777	1.7424	0.5222	3.5570	11.1132	1809.3387
1.0	0.0898	2.1045	0.3886	3.0780	10.5080	2018.3028

In Tables 1,2 and 3, $E(W)$ denotes the expected waiting time of an arbitrary customer. We can see that the $E(W)$ decreases as p increases upto some p_1 (shown in bold letters) and after that it increases. This is due to the effect of N-policy. As p increases (upto p_1), the number of customers in the system hit N more fast so that the server stops processing of service items and start serving customers and hence $E(W)$ decreases. When p becomes p_1, $E(W)$ starts increasing due to the diminished effect of N. Hence F_1 increases as p increases upto p_1 and after that it decreases. As we expect, $E(S)$ increases as p increases. As p increases, $E(it)$ decreases, since larger number of customers are served in a cycle. $E(ipo)$ increases upto p_1, as p increases. This is due to the effect of N-policy. As p increases, the number of customers in the system hit N more rapidly and hence customers leave the system quickly sothat the server can switch on to processing at a faster rate. When p increases beyond p_1, $E(ipo)$ decreases as p increases due to the diminished effect of N.

From Tables 1, 2 and 3, We can see that F_1 is strictly increasing on [0, p_1] and strictly decreasing on [p_1,1]. Thus,

1. If $F_1(p_1) \le 0$, then $F_1(p) \le 0$ for all $p \in [0,1]$. In this case, the maximum benefit is negative which implies that customers do not join the system even if there is no customer in the system.
2. If $F_1(0) > 0$ and $F_1(1) > 0$, then $F_1(p) > 0$ for all $p \in [0,1]$. In this case, customers prefer to join the system, because the minimal benefit is positive.
3. If $F_1(p_1) \ge 0$ and $F_1(0) < 0$, $\exists$ $p_e \in [0, p_1]$ such that $F_1(p_e) = 0$.
4. If $F_1(p_1) \ge 0$ and $F_1(1) < 0$, $\exists$ $p_e \in [p_1, 1]$ such that $F_1(p_e) = 0$.
5. If $F_1(p_1) \ge 0$, $F_1(0) < 0$ and $F_1(1) < 0$ then $\exists$ $p_e \in [0, p_1]$ such that $F_1(p_e) = 0$ and $p'_e \in [p_1, 1]$ such that $F_1(p'_e) = 0$.

Hence, if, either of the cases 3,4 and 5 happen, then customers are indifferent between joining and balking the system. Suppose that, case 3 holds. Then the above discussions imply that when the joining probability p adopted by other customers is greater than p_e, the expected net benefit of an arriving customer is positive provided he joins, thus the unique best response is 1. Conversely, the unique best response is 0 if $p < p_e$ because then the expected net benefit is negative. If $p = p_e$, every strategy is the best response since the expected net benefit is always 0. This behaviour illustrates a situation that an individual's best response is an increasing function of the strategy selected by other customers. Therefore, we expect a crowd situation in this case due to the effect of N-policy.

Next, suppose that, case 4 holds. Then the above discussions imply that when the joining probability p adopted by other customers is smaller than p_e, the expected net benefit of an arriving customer is positive provided he joins, thus the unique best response is 1. Conversely, the unique best response is 0 if $p > p_e$ because then the expected net benefit is negative. If $p = p_e$, every strategy is the best response since the expected net benefit is always 0. This behaviour illustrates a situation that an individual's best response is a decreasing function of the strategy selected by other customers. Therefore, we can avoid a crowd situation. This is due to the diminished effect of N-policy.

Next, suppose that case 5 holds, then the above discussions imply that when the joining probability p adopted by other customers is greater than p_e and less than p'_e, the expected net benefit of an arriving customer is positive provided he joins, thus the unique best response is 1. Conversely, the unique best response is 0 if $p < p_e$ or $p > p'_e$ because then the expected net benefit is negative. If $p = p_e$ or $p = p'_e$, every strategy is the best response since the expected net benefit is always 0.

Table 4. Individual optimum when $D_0 = (-20)$, $D_1 = (20)$.

$p \downarrow q \rightarrow$	10	20	30	40	50	60	70	75
0.02	-50.11	-60.11	-70.11	-80.11	-90.11	-100.11	-110.11	-120.11
0.1	39.78	29.78	19.78	9.78	-0.22	-10.22	-20.22	-30.22
0.2	52.14	42.14	32.14	22.14	12.14	2.14	-7.86	-17.86
0.3	56.19	46.19	36.19	26.19	16.19	6.19	-3.81	-13.81
0.4	58.16	48.16	38.16	28.16	18.16	8.16	-1.84	-11.84
0.5	59.31	49.31	39.31	29.31	19.31	9.31	-0.69	-10.69
0.6	60.04	50.04	40.04	30.04	20.04	10.04	0.04	-9.96
0.7	60.55	50.55	40.55	30.55	20.55	10.55	0.55	-9.45
0.8	60.90	50.90	40.90	30.90	20.90	10.90	0.90	-9.1
0.9	61.14	51.14	41.14	31.14	21.14	11.14	1.14	-8.86
1.0	61.29	51.29	41.29	31.29	21.29	11.29	1.29	-8.71

Table 5. Individual optimum when $D_0 = (-25)$, $D_1 = (25)$.

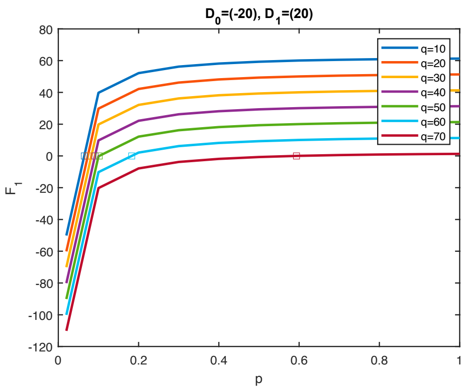
$p \downarrow q \rightarrow$	10	20	30	40	50	60	70	75
0.02	-25.12	-35.12	-45.12	-55.12	-65.12	-75.12	-85.12	-95.12
0.1	44.74	34.74	24.74	14.74	4.74	-5.26	-15.26	-25.26
0.2	54.58	44.58	34.58	24.58	14.58	4.58	-5.42	-15.42
0.3	57.77	47.77	37.77	27.77	17.77	7.77	-2.23	-12.23
0.4	59.31	49.31	39.31	29.31	19.31	9.31	-0.69	-10.69
0.5	60.19	50.19	40.19	30.19	20.19	10.19	0.19	-9.81
0.6	60.74	50.74	40.74	30.74	20.74	10.74	0.74	-9.26
0.7	61.08	51.08	41.08	31.08	21.08	11.08	1.08	-8.92
0.8	61.29	51.29	41.29	31.29	21.29	11.29	1.29	-8.71
0.9	61.36	51.36	41.36	31.36	21.36	11.36	1.36	-8.64
1.0	61.30	51.30	41.30	31.30	21.30	11.30	1.30	-8.70

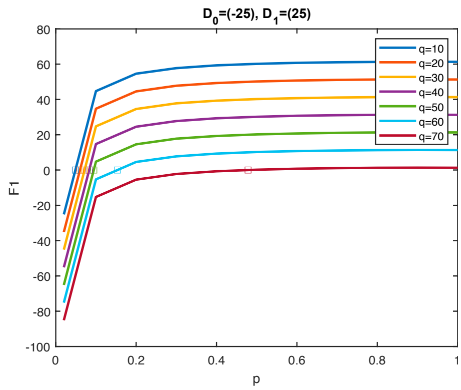
Table 6. Individual optimum when $D_0 = (-30),\ D_1 = (30).$

$p \downarrow q \rightarrow$	10	20	30	40	50	60	70	75
0.02	-8.46	-18.46	-28.46	-38.46	-48.46	-58.46	-68.46	-78.46
0.1	48.04	38.04	28.04	18.04	8.04	-1.96	-11.96	-21.96
0.2	56.19	46.19	36.19	26.19	16.19	6.19	-3.81	-13.81
0.3	58.80	48.80	38.80	28.80	18.80	8.80	-1.20	-11.20
0.4	60.05	50.05	40.05	30.05	20.05	10.05	0.05	-9.95
0.5	60.74	50.74	40.74	30.74	20.74	10.74	0.74	-8.26
0.6	61.14	51.14	41.14	31.14	21.14	11.14	1.14	-8.86
0.7	61.33	51.33	41.33	31.33	21.33	11.33	1.33	-8.67
0.8	61.34	51.34	41.34	31.34	21.34	11.34	1.34	-8.86
0.9	61.12	51.12	41.12	31.12	21.12	11.12	1.12	-8.88
1.0	60.51	50.51	40.51	30.51	20.51	10.51	0.51	-9.49

From Tables 4,5 and 6, we get the values of F_1 corresponding to different values of p and q when the arrival rates are 20, 25 and 30 respectively.

In our experiment, $\exists$ a q_1 such that $F_1(p_1) \geq 0,\ F_1(0) < 0, F_1(1) \geq 0$ and $\exists$ exactly one equillibrium p_e in $(0, p_1]$ for all $q \in [0, q_1)$ where $0 < q_1 < R$ (in Table 6, $q_1 = 70.51$). Also p_e is strictly increasing for all q in $[0, q_1)$ (in Figure 1, $(p_e, 0)$ corresponding to different q's are plotted using squares). This is due to the effect of N-policy. Also, $\exists$ a q_2, where $q_1 \leq q_2 < R$ such that when $q_1 \leq q \leq q_2$, $\exists\ p_e \in [0, p_1]$ and $p_e' \in [p_1, 1]$ such that p_e is strictly increasing and p_e' is strictly decreasing in $[q_1, q_2]$ (in Table 6, $q_2 = 71.34$). This case is shown in Figure 2. When $q \in (q_2, R],\ F_1(p) < 0$ for $p \in [0,1]$ and there is no equillibrium probability. Hence, if q increases (up to q_1), more customers are supposed to join the queue, since the server can start service only if the number of customers in the system hit N. When q increases from q_2 to R, customers do not join the system since the maximum benefit is negative.





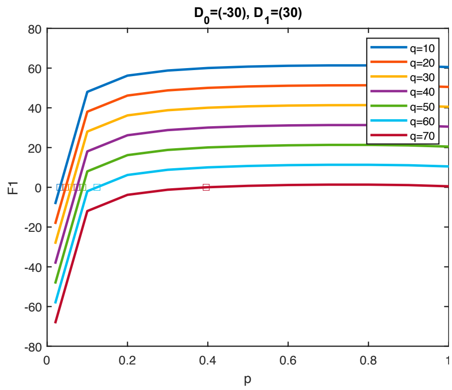


Figure 1. Effect of $q(<q_1)$ on individual equillibrium strategy.

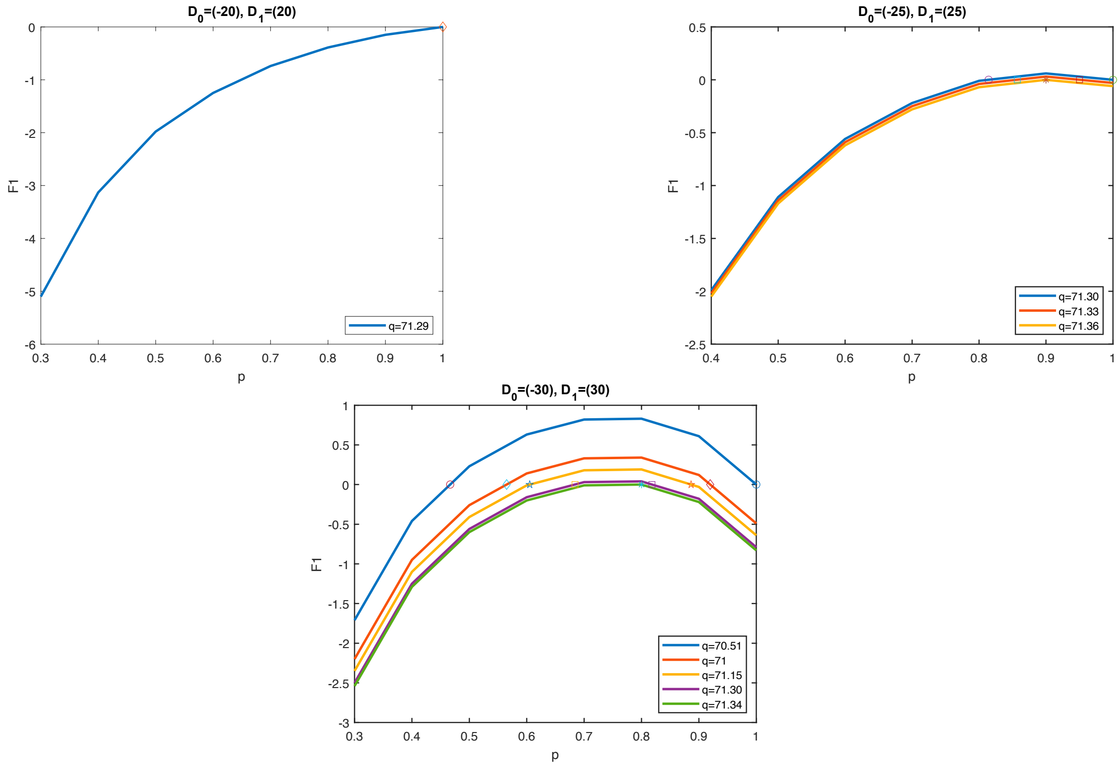


Figure 2. Effect of $q(q_1 \leq q \leq q_2)$ on individual equillibrium strategy.

Figure 1 shows individual equillibrium probabilities p_e as q varies ($0<q<q_1$), corresponding to different arrival rates. We can see that p_e increases as q increases for the three different arrival rates. But p_e decreases as arrival rate increases. Figure 2 shows individual equillibrium probabilities p_e, p_e' as q varies ($q_1 \leq q \leq q_2$) corresponding to different arrival rates. We see that p_e increases and p_e' decreases as q increases and coincides when $q = q_2$ for three different arrival rates.

Tables 7 and 8 show the effect of q on revenue of the server. Here, we see that F_2

decreases as q increases upto q_1. This happens because when q increases upto q_1, p_e increases and hence the rate of hitting N becomes faster so that $E(ipo)$ increases. But we see that when q increases in $[q_1, q_2]$, after a certain q-value, revenue function increases if p_e is the joining probability. This is due to the diminished effect of N-policy. Here, in all the cases, maximum revenue occur corresponding to $q = 10$ and the revenue decreases if a higher q is levied upto q_1. But when q increases beyond q_1, after a certain q-value, revenue increases if a higher q is levied.

Table 7. Revenue Maximization ($0 < q < q_1$).

q	$D_0 = (-20), D_1 = (20)$		$D_0 = (-25), D_1 = (25)$		$D_0 = (-30), D_1 = (30)$	
	p_e	F_2	p_e	F_2	p_e	F_2
10	0.0646	-15.31	0.0327	-11.22	0.0320	-12.45
20	0.0735	-16.82	0.0477	-14.45	0.0461	-16.08
30	0.0824	-18.32	0.0628	-17.67	0.0603	-19.67
40	0.0913	-19.82	0.0778	-20.81	0.0745	-23.20
50	0.1018	-21.56	0.0929	-23.92	0.0886	-26.65
60	0.1827	-34.50	0.1535	-35.91	0.1240	-35.01
70	0.5932	-84.23	0.4784	-84.63	0.3960	-84.29

Table 8. Revenue Maximization ($q_1 \le q \le q_2$).

D_0, D_1	q	p_e	F_2	p'_e	F_2
(-20), (20)	71.29($q_1 = q_2$)	1	-100.89	1	-100.89
(-25), (25)	71.30(q_1)	0.8143	-100.76	1	-91.78
	71.33	0.8571	-99.87	0.9500	-95.52
	71.36(q_2)	0.9000	-98.28	0.9000	-98.28
(-30), (30)	70.51(q_1)	0.4667	-92.10	1	-66.92
	71	0.5650	-98.98	0.9197	-80.85
	71.15	0.6053	-100.39	0.8864	-85.567
	71.30	0.6842	-100.66	0.8182	-93.25
	71.34(q_2)	0.8000	-94.85	0.8000	-94.85

Again, from Tables 1,2 and 3, we can see that F_3 increases as p inreases. But the rate of increase decreases as p increases. Here, the social optimum corresponds to $p = 1$ (p_s) in all cases.

Next, we vary values of N and L and see the effect of p on different performance measures. We fix $\alpha = \beta = \begin{bmatrix} 1 & 0 \end{bmatrix}$, $\gamma = \begin{bmatrix} 0.8 & 0.2 \end{bmatrix}$, $T = \begin{bmatrix} -50 & 50 \\ 0 & -50 \end{bmatrix}$, $S = \begin{bmatrix} -80 & 80 \\ 0 & -80 \end{bmatrix}$,

$$U=\begin{bmatrix}-150 & 150\\ 0 & -150\end{bmatrix}, R=75,\ q=60,\ h_w=50,\ h_1=2,\ h_2=1,\ c=30.$$

Table 9. Effect of p on different performances varying N and L.

(N,L)	p	$E(W)$	$E(S)$	$E(it)$	$E(ipo)$	F_1
(3,2)	0.1	0.5045	1.0248	1.8794	0.6445	-10.2243
	0.2	0.2571	1.0514	1.7601	1.2429	2.1430
	0.3	0.1762	1.0806	1.6421	1.7921	6.1878
	0.4	0.1368	1.1129	1.5254	2.2882	8.1599
	0.5	0.1138	1.1486	1.4102	2.7271	9.3077
(3,3)	0.1	0.4888	1.0241	2.8168	0.6562	-9.4391
	0.2	0.2433	1.0487	2.6346	1.2652	2.8337
	0.3	0.1642	1.0746	2.4528	1.8402	6.7919
	0.4	0.1263	1.1028	2.2713	2.3687	8.6865
	0.5	0.1047	1.1341	2.0905	2.8433	9.7656
(3,4)	0.1	0.4880	1.0240	3.8110	0.6506	-9.4013
	0.2	0.2421	1.0481	3.6104	1.2686	2.8936
	0.3	0.1628	1.0728	3.3952	1.8514	6.8617
	0.4	0.1249	1.0988	3.1640	2.3942	8.7560
	0.5	0.1035	1.1271	2.9182	2.8892	9.8269
(4,3)	0.1	0.7456	1.5143	2.8244	0.4878	-22.2785
	0.2	0.3761	1.5217	2.6469	0.9567	-3.8066
	0.3	0.2556	1.5286	2.4667	1.4062	2.2207
	0.4	0.1968	1.5389	2.2847	1.8296	5.1614
	0.5	0.1624	1.5550	2.1023	2.2174	6.8805
(4,4)	0.1	0.7301	1.5151	3.7633	0.4919	-21.5035
	0.2	0.3626	1.5217	3.5220	0.9736	-3.1325
	0.3	0.2438	1.5282	3.2756	1.4414	2.8079
	0.4	0.1866	1.5391	3.0260	1.8842	5.6683
	0.5	0.1521	1.5585	3.6070	2.3134	7.3953
(5,4)	0.1	0.9844	2.0093	3.7704	0.3916	-34.2183
	0.2	0.4924	2.0047	3.5349	0.7743	-9.6201
	0.3	0.3324	1.9939	3.2917	1.1511	-1.6189
	0.4	0.2546	1.9837	3.0421	1.5163	2.2700
	0.5	0.2093	1.9793	2.7891	1.8599	4.5362
(5,5)	0.1	0.9689	2.011	4.7101	0.3947	-33.4434
	0.2	0.4792	2.0051	4.4102	0.7881	-8.9589
	0.3	0.3210	1.9937	4.0986	1.1806	-1.0519
	0.4	0.2450	1.9853	3.7787	1.5617	2.7517
	0.5	0.2013	1.9855	3.4557	1.9168	4.9373

From Table 9, we can see that when N increases (keeping L fixed), $E(W)$ increases. This happens because as N increases, the number of customers in the system hits N at a slower rate and thus the server switches to service mode at a slower rate. When L increases (keeping N fixed), $E(W)$ decreases. This is because when L increases, more customers get service according to $PH(\gamma, U)$ whose service rate is greater than that of $PH(\beta, S)$. As a result, when N increases (keeping L fixed) F_1 decreases and when L increases (keeping N fixed), F_1 increases.

When N increases (keeping L fixed), $E(S)$ increases as expected. As N increases (keeping L fixed), $E(it)$ increases since lesser number of customers are served in a cycle. When L increases (keeping N fixed), $E(it)$ increases, since more items are processed in a cycle. $E(ipo)$ decreases as N increases since the server switches to service mode at a slower rate and as a result to inventory processing at a slower rate.

Next, we consider the following two sets of matrices for D_0 and D_1.

1. MAP with positive correlation (MPA)

$$D_0 = \begin{bmatrix} -27.0697 & 27.0697 & 0 \\ 0 & -45.0981 & 0 \\ 0 & 0 & -595.5331 \end{bmatrix}, \quad D_1 = \begin{bmatrix} 0 & 0 & 0 \\ 42.3911 & 0 & 2.7070 \\ 189.4878 & 0 & 406.0453 \end{bmatrix}$$

2. MAP with negative correlation (MNA)

$$D_0 = \begin{bmatrix} -7.9382 & 7.9382 & 0 \\ 0 & -13.2250 & 0 \\ 0 & 0 & -404.0521 \end{bmatrix}, \quad D_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0.7938 & 0 & 12.4311 \\ 119.0723 & 0 & 284.9797 \end{bmatrix}$$

These two MAP processes are normalized so as to have an arrival rate of 20. The arrival process labeled MNA has correlated arrivals with correlation between two successive interarrival times given by -0.1368 and the arrival process corresponding to the one labelled MPA has a positive correlation with value 0.1368.

We fix other parameters same as in the case of Poisson arrivals.

Table 10. Effect of p on various performance measures, when arrival follows MPA.

p	$E(W)$	$E(S)$	$E(it)$	$E(ipo)$	F_1
0.1	0.0905	1.0695	1.7309	1.3096	10.4728
0.2	0.0931	1.1486	1.5323	2.1580	10.3428
0.3	0.1002	1.2185	1.3794	2.7405	9.9915
0.4	0.1060	1.2807	1.2585	3.1396	9.7006
0.5	0.1108	1.3371	1.1609	3.4116	9.4579

Table 11. Effect of p on various performance measures, when arrival follows MNA.

p	$E(W)$	$E(S)$	$E(it)$	$E(ipo)$	F_1
0.1	0.1657	0.7477	1.7519	0.9167	6.7171
0.2	0.1148	0.8068	1.5702	1.3605	9.2616
0.3	0.1116	0.9362	1.4262	1.7000	9.4198
0.4	0.1140	1.0727	1.3100	1.9477	9.2978
0.5	0.1181	1.2100	1.2148	2.1222	9.0960

From Table 10, we can see that the $E(W)$ increases as p increases, different from the case of Poisson arrivals. This is due to the effect of positive correlation. Hence, F_1 decreases as p increases. But in the case of MNA (Table 11), $E(W)$ decreases as p increases upto some p_1 (here $p_1 = 0.3$) and after that it increases, as in the case of Poisson arrivals and hence F_1 increases as p increases upto p_1 and after that it decreases. In both cases, as we expect, $E(S)$ increases as p increases. As p increases, $E(it)$ decreases, since larger number of customers are served in a cycle. $E(ipo)$ increases as p increases. This is because as p increases, the number of customers in the system hit N more rapidly and hence customers leave the system quickly sothat the server can switch on to processing at a faster rate.

5. Special Case: The System in Normal Mode

5.1. Waiting time analysis

To find the waiting time of a customer who joins for service at an epoch in the long run, we have to consider different possibilities depending on the status of server at that time. Let Ev denote the event the system is working in normal mode. Let $W(t \mid Ev)$ be the conditional waiting time of a customer who arrives at time t and $W^*(s \mid Ev)$ be the corresponding conditional LST.

Let $w_{h,i,j,k,l}$ denote the probabaility that the tagged customer finds the system in the state (h,i,j,k,l) immedietly after his arrival when the system is in normal mode.

Then

$$w_{h,0,1,k,l} = \sum_{l'=1}^{n} \frac{p d_{l'l}^{(1)}}{-d_{l'l'}^{(0)} - (1-p)\delta_{l'} - S_{kk}} x_{h-1,0,1,k,l'}, \quad 2 \le h \le N-1, \text{ or } h \ge N+1, 1 \le k \le m_2, 1 \le l \le n.$$

$$w_{N,0,1,k,l} = \sum_{l'=1}^{n} \frac{p d_{l'l}^{(1)}}{-d_{l'l'}^{(0)} - (1-p)\delta_{l'} - S_{kk}} x_{N-1,0,1,k,l'}, \quad 1 \le k \le m_2, 1 \le l \le n.$$

$$w_{h,i,1,k,l}=\sum_{l'=1}^{n}\frac{pd_{l'l}^{(1)}}{-d_{l'l'}^{(0)}-(1-p)\delta_{l'}-U_{kk}}x_{h-1,i,1,k,l'},\quad 2\le h\le N-1,\,1\le L-N+h-1,$$

$$1\le k\le m_3,\,1\le l\le n.$$

$$w_{h,i,1,k,l}=\sum_{l'=1}^{n}\frac{pd_{l'l}^{(1)}}{-d_{l'l'}^{(0)}-(1-p)\delta_{l'}-U_{kk}}x_{h-1,i,1,k,l'},\quad h\ge N+1,\,1\le i\le L,\,1\le k\le m_3,$$

$$1\le l\le n.$$

$$w_{N,i,1,k,l}=\sum_{l'=1}^{n}\frac{pd_{l'l}^{(1)}}{-d_{l'l'}^{(0)}-(1-p)\delta_{l'}-S_{kk}}x_{N-1,i,1,k,l'},\quad 1\le i\le L, 1\le k_2\le m_2, 1\le l\le n.$$

Case I: $L\le N$

Case (1)

Let Ev_1 denote the event that the tagged customer immedietly after his arrival finds the system in the state $(r+1,0,1,k,l)$, where $r\ge 1,\,1\le k\le m_2,\,1\le l\le n$. In this case, processed item is not available to any customer. Thus waiting time is the sum of residual service time and r service time each following $PH(\beta,S)$.

$$W^*(s\mid Ev,Ev_1)=e_u'(sI-S)^{-1}S^0(\beta(sI-S)^{-1}S^0)^r.$$

Case (2)

Let Ev_2 denote the event that the tagged customer immedietly after his arrival finds the system in the state $(r+1,i,1,k,l)$, where $1\le r\le N-1,\,1\le i\le L-N+r$, $1\le k\le m_3,\,1\le l\le n$. In this case, processed item is available to i customers. Thus waiting time is the sum of residual service time and $i-1$ service time each following $PH(\gamma,U)$ and $r+1-i$ service time each following $PH(\beta,S)$.

$$W^*(s\mid Ev,Ev_2)=e_u'(sI-U)^{-1}U^0(\gamma(sI-U)^{-1}U^0)^{i-1}(\beta(sI-S)^{-1}S^0)^{r+1-i}.$$

Case (3)

Let Ev_3 denote the event that the tagged customer immedietly after his arrival finds the system in the state $(r+1,i,1,k,l)$, where $r\ge N,\,1\le i\le L,\,1\le k\le m_3,\,1\le l\le n$. In this case, processed item is available to i customers. Thus waiting time is the sum of residual service time and $i-1$ service time each following $PH(\gamma,U)$ and $r+1-i$ service time each following $PH(\beta,S)$.

$$W^*(s\mid Ev,Ev_3)=e_u'(sI-U)^{-1}U^0(\gamma(sI-U)^{-1}U^0)^{i-1}(\beta(sI-S)^{-1}S^0)^{r+1-i}.$$

Thus the conditional LST of the waiting time,

$$W^*(s\mid Ev)=\frac{1}{d_3}\Big[\sum_{r=1}^{\infty}\sum_{k=1}^{m_2}\sum_{l=1}^{n}W^*(s\mid Ev,Ev_1)w_{r+1,0,1,k,l}+\sum_{r=1}^{N-1}\sum_{i=1}^{L-N+r}\sum_{k=1}^{m_3}\sum_{l=1}^{n}W^*(s\mid Ev,Ev_2)w_{r+1,i,1,k,l}$$

$$+\sum_{r=N}^{\infty}\sum_{i=1}^{L}\sum_{k=1}^{m_3}\sum_{l=1}^{n}W^*(s\mid Ev,Ev_3)w_{r+1,i,1,k,l}\Big],\tag{10}$$

where

$$d_3 = \sum_{r=1}^{\infty}\sum_{k=1}^{m_2}\sum_{l=1}^{n} w_{r+1,0,1,k,l} + \sum_{r=1}^{N-1}\sum_{i=1}^{L-N+r}\sum_{k=1}^{m_3}\sum_{l=1}^{n} w_{r+1,i,1,k,l} + \sum_{r=N}^{\infty}\sum_{i=1}^{L}\sum_{k=1}^{m_3}\sum_{l=1}^{n} w_{r+1,i,1,k,l}. \tag{11}$$

Case II: $L > N$

Case (1)

Let Fv_1 denote the event that the tagged customer immedietly after his arrival finds the system in the state $(r+1,0,1,k,l)$, where $r \geq 1, 1 \leq k \leq m_2, 1 \leq l \leq n$. In this case, processed item is not available to any customer. Thus waiting time is the sum of residual service time and r service time each following $PH(\beta, S)$.

$$W^*(s \mid Ev, Fv_1) = e'_u(sI - S)^{-1} S^0 (\beta(sI - S)^{-1} S^0)^r.$$

Case (2)

Let Fv_2 denote the event that the tagged customer immedietly after his arrival finds the system in the state $(r+1,i,1,k,l)$, where $1 \leq r \leq N-1$, $1 \leq i \leq L-N+r$, $1 \leq k \leq m_3$, $1 \leq l \leq n$.

Case (i), $1 \leq i < r+1$

In this case, processed item is available to i customers. Thus the conditional LST,

$$W^*(s \mid Ev, Fv_2) = e'_u(sI - U)^{-1} U^0 (\gamma(sI - U)^{-1} U^0)^{i-1} (\beta(sI - S)^{-1} S^0)^{r+1-i}.$$

Case (ii), $r+1 \leq i \leq L-N+r$

In this case, processed item is available to all the $r+1$ customers. Thus the conditional LST,

$$W^*(s \mid Ev, Fv_2) = e'_u(sI - U)^{-1} U^0 (\gamma(sI - U)^{-1} U^0)^r.$$

Case (3)

Let Fv_3 denote the event that the tagged customer immedietly after his arrival finds the system in the state $(r+1,i,1,k,l)$, where $r \geq N, 1 \leq i \leq L, 1 \leq k \leq m_3, 1 \leq l \leq n$.

Case (i), $N \leq r \leq L$

Case (a), $1 \leq i < r+1$

In this case, processed item is available to i customers. Thus the conditional LST,

$$W^*(s \mid Ev, Fv_3) = e'_u(sI - U)^{-1} U^0 (\gamma(sI - U)^{-1} U^0)^{i-1} (\beta(sI - S)^{-1} S^0)^{r+1-i}.$$

Case (b), $r+1 \leq i \leq L$

In this case, processed item is available to all the $r+1$ customers. Thus the conditional LST,

$$W^*(s \mid Ev, Fv_3) = e'_u(sI - U)^{-1} U^0 (\gamma(sI - U)^{-1} U^0)^r.$$

Case (ii), $r \geq L+1$

In this case, processed item is available to i customers. Thus the conditional LST,

$$W^*(s \mid Ev, Fv_3) = e'_u(sI - U)^{-1} U^0 (\gamma(sI - U)^{-1} U^0)^{i-1} (\beta(sI - S)^{-1} S^0)^{r+1-i}.$$

Thus the conditional LST of the waiting time,

$$W^*(s\mid Ev)=\frac{1}{d_4}[\sum_{r=1}^{\infty}\sum_{k=1}^{m_2}\sum_{l=1}^{n}W^*(s\mid Ev,Fv_1)w_{r+1,0,1,k,l}+\sum_{r=1}^{N-1}\sum_{i=1}^{L-N+r}\sum_{k=1}^{m_3}\sum_{l=1}^{n}W^*(s\mid Ev,Fv_2)w_{r+1,i,1,k,l}$$
$$+\sum_{r=N}^{\infty}\sum_{i=1}^{L}\sum_{k=1}^{m_3}\sum_{l=1}^{n}W^*(s\mid Ev,Fv_3))w_{r+1,i,1,k,l}], \qquad (12)$$

where

$$d_4=\sum_{r=1}^{\infty}\sum_{k=1}^{m_2}\sum_{l=1}^{n}w_{r+1,0,1,k,l}+\sum_{r=1}^{N-1}\sum_{i=1}^{L-N+r}\sum_{k=1}^{m_3}\sum_{l=1}^{n}w_{r+1,i,1,k,l}+\sum_{r=N}^{\infty}\sum_{i=1}^{L}\sum_{k=1}^{m_3}\sum_{l=1}^{n}w_{r+1,i,1,k,l}\,. \qquad (13)$$

We fix $N=3,\ L=2,\ \alpha=\beta=\begin{bmatrix}1 & 0\end{bmatrix},\ \gamma=\begin{bmatrix}0.8 & 0.2\end{bmatrix},\ T=\begin{bmatrix}-50 & 50\\ 0 & -50\end{bmatrix},$

$S=\begin{bmatrix}-80 & 80\\ 0 & -80\end{bmatrix},\ U=\begin{bmatrix}-150 & 150\\ 0 & -150\end{bmatrix},\ R=75,\ q=60,\ h_w=50,\ h_1=2,\ h_2=1,\ c=30.$

Table 12. Effect of p on various performance measures, when $D_0=(-20),\ D_1=(20)$.

p	$E(W)$	$E(S)$	$E(it)$	$E(ipo)$	F_1	F_3
0.1	0.0485	1.5214	0.6972	19.3813	12.5747	-440.0284
0.2	0.0499	1.5604	0.6564	18.3266	12.5034	-263.5605
0.3	0.0512	1.6081	0.6117	17.2386	12.4386	-86.3546
0.4	0.0524	1.6657	0.5640	16.1240	12.3798	91.4217
0.5	0.0535	1.7341	0.5145	14.9939	12.3259	269.4597
0.6	0.0545	1.8144	0.4645	13.8595	12.2749	447.4200
0.7	0.0555	1.9081	0.4153	12.7308	12.2237	624.9757
0.8	0.0566	2.0174	0.3677	11.6154	12.1681	801.8239
0.9	0.0579	2.1455	0.3224	10.5189	12.1027	977.6669
1.0	0.0596	2.2971	0.2796	9.4448	12.0200	1152.1810

From Table 12, we see that $E(W)$ increases as p increases. This happens since when the system is working in normal mode, the number of customers accumulating in the system increases with increasing value of p. As p increases F_1 decreases consequent to increase in $E(W)$. As we expect, $E(S)$ increases as p increases. As p increases, $E(it)$ decreases, since larger number of customers get served in a cycle. $E(ipo)$ decreases as p increases. This happens due to the fact that when p increases more customers accumulate in the system and hence customers leave the system slowly so that the server switch on to processing at a slower rate. Also from table 12, we see that F_3 increases as p increases. Thus the social optimum corresponds to $p=1$.

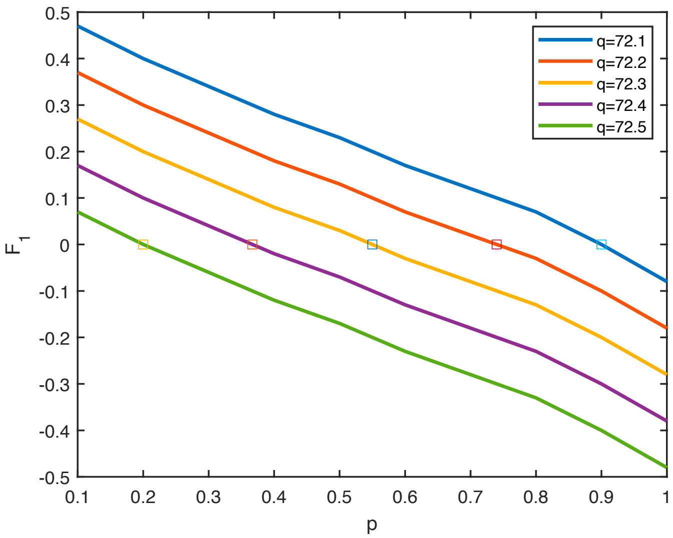


Figure 3. Effect of q on individual equillibrium strategy when $D_0 = (-20), D_1 = (20)$.

Here when $q < 72.02$, the expected net benefit is always positive. When q increases beyond 72.02, (see Table 12), we can find a $p_e \in [0,1]$ such that $F_1(p_e) = 0$ and p_e is decreasing (see Figure 3). Here when the joining probability p adopted by other customers is smaller than p_e, the expected net benefit of an arriving customer is positive provided he joins. Thus the unique best response is 1. Conversely, the unique best response is 0 if $p > p_e$ since, the expected net benefit is negative. If $p = p_e$, every strategy is the best response since the expected net benefit is always 0. This behaviour illustrates a situation that an individuals best response is a decreasing function of the strategy selected by other customers. Therefore, we can avoid a crowd situation.

Table 13. Effect of q on Revenue function.

q	p_e	F_2
72.1	0.9000	-302.1809
72.2	0.7400	-357.9769
72.3	0.5500	-425.8322
72.4	0.3667	-491.4621
72.5	0.2000	-549.3741

Also, in this case revenue function F_2 decreases as q increases. This happens due to the fact that as q increases, the equillibrium probability p_e decreases and hence $E(ipo)$ increases (see Table 13).

6. Conclusion

In this paper, we considered a MAP/PH/1 queue with processing of service items under Vacation and N-policy with impatient customers. We studied this system as a Level Dependent QBD. We explained how to find the steady state vectors of the system approximately by finite truncation method. We found the distribution of time until the number of customers hit N. Several system performance characteristics were computed. We computed LST of the waiting time distribution for the case of no reneging. Also we performed some numerical experiments for computing individual optimal strategy, maximum revenue to the server and social optimal strategy for the special case of no reneging. We discussed the special case in which the system is in normal mode. In this case also, we computed LST of the waiting time distribution and performed some numerical experiments for computing individual optimal strategy, maximum revenue to the server and social optimal strategy.

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