Queueing Models and
Service Management
Vol. 9, No. 1, page 1–43, 2026

Stochastic Fluid Flows with Upward Jumps and Phase Transitions: Analysis Through Matrix-Analytic Methods

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(*Received April 2024; accepted September 2025*)

Abstract: We consider a stochastic fluid model $\{(X(t), J(t)) : t \geq 0\}$ with level variable $X(t) \geq 0$, phase variable $J(t)$ and some fixed 'jump' levels $q_N > \ldots > q_1 > 0$. The process is driven by a continuous-time Markov chain $\{J(t) : t \geq 0\}$ with state space $\mathcal{S}$, generator $\mathbf{T}$, and real-valued fluid rates $c_i \in \mathbb{R}$ for all $i \in \mathcal{S}$. The evolution of the level variable $X(t)$ is such that $dX(t)/dt = c_{J(t)}$ whenever $X(t) > 0$, and as soon as the process hits $X(t) = 0$, the phase variable $J(t)$ transitions to some phase in $\mathcal{S}$, while the level variable $X(t)$ may jump to some level $q_n > 0$, remain at the boundary $q_0 = 0$ or reflect from it, and a phase transition may also occur. The process was previously analysed using various algebraic methods in a special case with $N = 1$ and without special behaviour at the boundary q_0. Here, for the first time, we analyse this process using matrix-analytic methods, which is a powerful methodology in the field of applied probability, suitable for convenient numerical analysis. We present methodology for the computation of the stationary and transient distribution of the key performance measures of the model under general assumptions and illustrate the theory with numerical examples.

Keywords: Markov chains, matrix-analytic methods, stationary distribution, stochastic fluid models, stochastic fluid models with jumps, transient analysis.

1. Introduction

Matrix-analytic methods (MAMs) is the theory within the field of applied probability that is a rich source of stochastic models and theoretical results for the mathematical analysis of a wide range of real-world systems evolving in random environments, and powerful

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algorithms for the efficient numerical computation of their performance measures. For the overview of the key mathematical objects and results within the MAMs literature the reader is referred to Neuts [24], Latouche and Ramaswami [21], Bini, Latouche and Meini [10], He [15], and Bladt and Nielsen [11]. Two fundamental, closely related classes of Markovian-modulated models within the theory of MAMs are Quasi-Birth-and-Death processes (QBDs) and stochastic fluid models (SFMs), in which the evolution of the level variable $X(t)$ (discrete in QBDs / continuous in SFMs) is driven by the evolution of the phase variable $J(t)$, see e.g. Bean, Latouche and Taylor [3], Bean and O'Reilly [4], Da Silva Soares [12], and Latouche and Nguyen [20]. For the review of the methodology and application examples in queueing theory, such as multi-server queueing systems, ride-hailing platforms, organ transplantation systems, perishable inventory systems, and risk models in finance, the reader is also referred to Wu [29], and He and Wu [16, 17].

Here, we apply this powerful methodology for the stationary and transient analysis of a class of SFMs in which the level variable jumps to some level $q > 0$, immediately after hitting level zero from above. This class of models has application potential to a wide range of problems in real-world, since continuous-time Markov chains are ubiquitous in applications, and so SFMs can be used to analyse performance measures of many real-world systems. SFMs with jumps further extend the application potential of SFMs.

Related SFMs were previously studied in Kulkarni and Yan in [19] and by Nabli and Abdallah in [23] using spectral methods, and in Bean, O'Reilly and Sargison [7] using MAMs. The advantage of the MAMs methodology applied here is that the analysis of these models involves meaningful physical interpretations and conditioning on the evolution of the sample paths, which not only enables convenient proof techniques, but also results in efficient computational methods that follow from these interpretations.

We build upon the methodology for the direct analysis of the SFMs and the physical interpretations of the *fluid generators* derived by Bean, O'Reilly and Taylor in [8, 9], Bean and O'Reilly in [5], and Samuelson, Bean and O'Reilly in [27]. In our approach we apply convenient arguments from the classic theory of MAMs [21, 24] in which we condition on hitting certain levels within the sample paths, and then represent the quantities corresponding to the related parts of the sample paths in terms of these generators.

Consider a stochastic fluid model (SFM) $\{(X(t), J(t)) : t \geq 0\}$ with thresholds $q_N > \ldots > q_1 > 0$, $q_0 = 0$, level variable $X(t) \geq 0$, phase variable $J(t) \in \mathcal{S}$, generator $\mathbf{T}$, and real rates $c_{J(t)} \in \mathbb{R}$ such that

- $\{J(t) : t \geq 0\}$ evolves according to an irreducible continuous-time Markov Chain (CTMC) with some finite state space $\mathcal{S} = \{1, \ldots, m\}$ and generator $\mathbf{T} = [T_{ij}]_{i,j \in \mathcal{S}}$, where we partition $\mathcal{S} = \mathcal{S}_+ \cup \mathcal{S}_- \cup \mathcal{S}_0$ according to phases with positive, negative and zero rates $c_{J(t)}$ respectively, $\mathcal{S}_+ = \{i : c_i > 0\}$, $\mathcal{S}_- = \{i : c_i < 0\}$, $\mathcal{S}_0 = \{i : c_i = 0\}$,

- when $X(t) > 0$ then the level $X(t)$ changes at rate $c_{J(t)}$, and
- as soon as the process hits $X(t) = 0$, which occurs in some phase $J(t)$ with $c_{J(t)} < 0$, then one of the following alternatives may occur:
  - the phase process makes a transition to some phase in $j \in \mathcal{S}$ with probability $P_{ij}^{(q_0)}$ being an element of the matrix $\mathbf{P}^{(q_0)} = [P_{ij}^{(q_0)}]_{i\in\mathcal{S}_-, j\in\mathcal{S}}$, or
  - the level $X(t)$ instanteneously jumps to level $q_n > 0$ for some $n = 1, \ldots, N$, and the phase process makes a transition to some phase $j \in \mathcal{S}$ with probability $P_{ij}^{(q_n)}$ being an element of the matrix $\mathbf{P}^{(q_n)} = [P_{ij}^{(q_n)}]_{i\in\mathcal{S}_-, j\in\mathcal{S}}$,

  where we assume that $\sum_{n=0}^{N} \mathbf{P}^{(q_n)}\mathbf{1} = \mathbf{1}$.

We partition generator $\mathbf{T}$ according to the partitioning of $\mathcal{S}$,

$$\mathbf{T} = \begin{bmatrix} \mathbf{T}_{++} & \mathbf{T}_{+-} & \mathbf{T}_{+0} \\ \mathbf{T}_{-+} & \mathbf{T}_{--} & \mathbf{T}_{-0} \\ \mathbf{T}_{0+} & \mathbf{T}_{0-} & \mathbf{T}_{00} \end{bmatrix}. \tag{1}$$

We denote $\mathbf{C} = \text{diag}(c_i)_{i\in\mathcal{S}}$, $\mathbf{C}_+ = \text{diag}(c_i)_{i\in\mathcal{S}_+}$ and $\mathbf{C}_- = \text{diag}(c_i)_{i\in\mathcal{S}_-}$, and define the *fluid* generators $\mathbf{Q}(s)$ and $\mathbf{Q} = \mathbf{Q}(0)$ introduced for standard SFMs without jumps by Bean, O'Reilly and Taylor in [8], such that

$$\mathbf{Q}(s) = \begin{bmatrix} \mathbf{Q}_{++}(s) & \mathbf{Q}_{+-}(s) \\ \mathbf{Q}_{-+}(s) & \mathbf{Q}_{--}(s) \end{bmatrix}, \tag{2}$$

with

$$\begin{aligned} \mathbf{Q}_{++}(s) &= (\mathbf{C}_+)^{-1}(\mathbf{T}_{++} - s\mathbf{I} - \mathbf{T}_{+0}(\mathbf{T}_{00} - s\mathbf{I})^{-1}\mathbf{T}_{0+}), \\ \mathbf{Q}_{+-}(s) &= (\mathbf{C}_+)^{-1}(\mathbf{T}_{+-} - \mathbf{T}_{+0}(\mathbf{T}_{00} - s\mathbf{I})^{-1}\mathbf{T}_{0-}), \\ \mathbf{Q}_{-+}(s) &= (|\mathbf{C}_-|)^{-1}(\mathbf{T}_{-+} - \mathbf{T}_{-0}(\mathbf{T}_{00} - s\mathbf{I})^{-1}\mathbf{T}_{0+}), \\ \mathbf{Q}_{--}(s) &= (|\mathbf{C}_-|)^{-1}(\mathbf{T}_{--} - s\mathbf{I} - \mathbf{T}_{-0}(\mathbf{T}_{00} - s\mathbf{I})^{-1}\mathbf{T}_{0-}). \end{aligned} \tag{3}$$

A useful physical interpretation of $\mathbf{Q}$ established in [8, 9] is that $[e^{\mathbf{Q}y}]_{ij}$ records the probability of observing phase j at the time $\omega(y) = \inf\{t > 0 : \int_0^t |c_{J(u)}|du = y\}$ when the total amount of fluid that has entered the buffer reaches y, in a related SFM $\{(\widehat{X}(t), J(t)) : t \geq 0\}$ with nonnegative rates $|c_{J(t)}|$, given the initial condition of starting from phase i at time 0; and $[e^{\mathbf{Q}(s)y}]_{ij}$ is the corresponding Laplace-Stieltjes transform (LST) $\mathbb{E}(e^{-s\omega(y)}1\{J(\omega(y)) = j\} \mid J(0) = i)$ of the time to do so. That is, $\mathbf{Q}(s)$ and $\mathbf{Q}$ are generators with respect to the fluid level (rather than time), and so this is why we refer to them as the fluid generators.

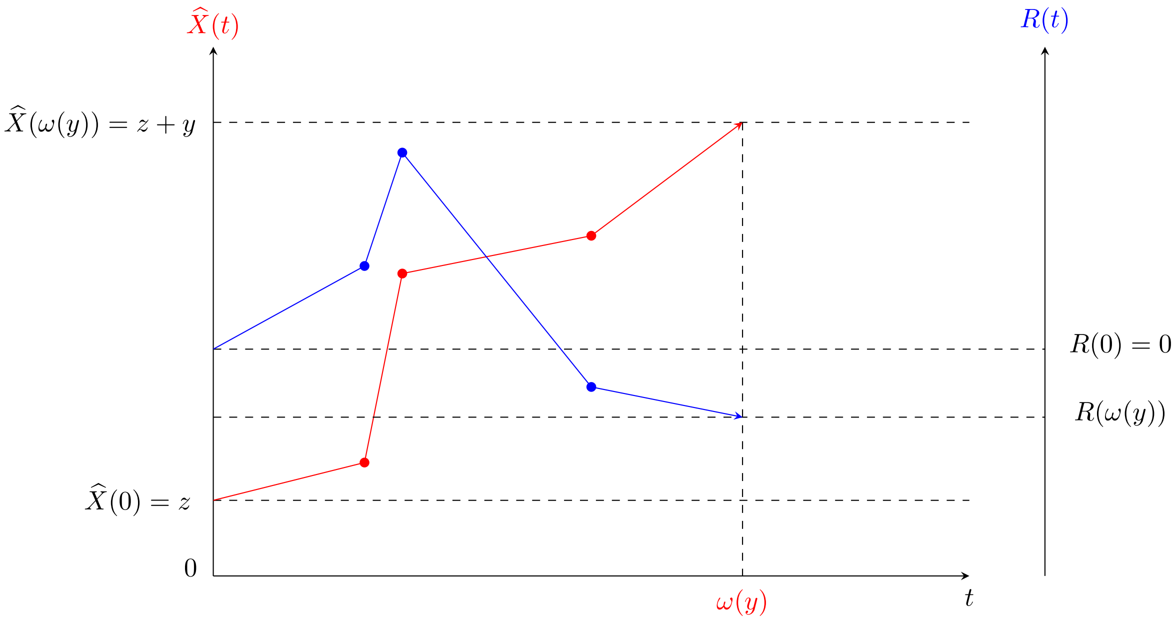

Figure 1. Consider SFM $\{(\widehat{X}(t), J(t)) : t \geq 0\}$ with rates $|c_{J(t)}|$. Then $\omega(y)$ is the first time at which the level $\widehat{X}(\cdot)$ hits $z + y$, given start from some $\widehat{X}(0) = z \geq 0$ at time zero. The total reward/cost $R(\omega(y))$ accumulated at time $\omega(y)$ is interpreted as the level $R(\omega(y))$ at time $\omega(y)$ given $R(0) = 0$, in a *doubly-unbounded* SFM $\{(R(t), J(t)) : t \geq 0\}$ with $R(t) \in (-\infty, +\infty)$ and some real-valued rates $r_{J(t)}$ (and no jumps).

We will also apply one of the fluid generators described in [5, 27], given by

$$\mathbf{W}(s) = \left[\begin{array}{cc} \mathbf{W}_{++}(s) & \mathbf{W}_{+-}(s) \\ \mathbf{W}_{-+}(s) & \mathbf{W}_{--}(s) \end{array} \right], \tag{4}$$

with

$$\begin{aligned} \mathbf{W}_{++}(s) &= (\mathbf{C}_+)^{-1}(\mathbf{T}_{++} - s\mathbf{R}_+ - \mathbf{T}_{+0}(\mathbf{T}_{00} - s\mathbf{R}_0)^{-1}\mathbf{T}_{0+}), \\ \mathbf{W}_{+-}(s) &= (\mathbf{C}_+)^{-1}(\mathbf{T}_{+-} - \mathbf{T}_{+0}(\mathbf{T}_{00} - s\mathbf{R}_0)^{-1}\mathbf{T}_{0-}), \\ \mathbf{W}_{-+}(s) &= (|\mathbf{C}_-|)^{-1}(\mathbf{T}_{-+} - \mathbf{T}_{-0}(\mathbf{T}_{00} - s\mathbf{R}_0)^{-1}\mathbf{T}_{0+}), \\ \mathbf{W}_{--}(s) &= (|\mathbf{C}_-|)^{-1}(\mathbf{T}_{--} - s\mathbf{R}_- - \mathbf{T}_{-0}(\mathbf{T}_{00} - s\mathbf{R}_0)^{-1}\mathbf{T}_{0-}), \end{aligned} \tag{5}$$

where $\mathbf{R}_+ = \mathrm{diag}(r_i)_{i\in\mathcal{S}_+}$, $\mathbf{R}_- = \mathrm{diag}(r_i)_{i\in\mathcal{S}_-}$, $\mathbf{R}_0 = \mathrm{diag}(r_i)_{i\in\mathcal{S}_0}$ record real-valued rates $r_i \in \mathbb{R}$ at which some reward/cost accumulates when $J(t) = i$.

The physical interpretation of $\mathbf{W}(s)$ is that $[e^{\mathbf{W}(s)y}]_{ij}$ records the LST $\mathbb{E}(e^{-sR(\omega(y))}1\{J(\omega(y)) = j\} \mid J(0) = i)$ of the total reward/cost $R(t) = \int_{u=0}^{t} r_{J(u)}du$ accumulated at the time $t = \omega(y)$ and doing so in phase $J(\omega(y)) = j$, given the initial condition of starting from phase i at time 0. Further, $\mathbf{W}(0) = \mathbf{Q}$, and if $r_i = 1$ for all $i \in \mathcal{S}$, then we have

$\mathbf{W}(s) = \mathbf{Q}(s)$. That is, $\mathbf{W}(s)$ is a generalisation of $\mathbf{Q}(s)$. Note that since $R(t) \in (-\infty, +\infty)$, we interpret it as the level variable in an unbounded SFM with rates r_i, driven by the phase process $J(t)$. We illustrate this in Figure 1.

The rest of the paper is organised as follows. In Section 2 we state the key results for the standard SFMs without jumps, and also derive some new results using direct methods of analysis within MAMs, based on the application of the fluid generators. We then build on this approach in the sections that follow, and consider the general SFMs with upward jumps and analyse them using MAMs. We focus on the stationary and the transient analysis in Sections 3 and 4, respectively. In Section 5 we illustrate the application of the theory with numerical examples. This is followed by concluding remarks in Section 6.

Throughout we assume that $\mu = \sum_{i \in \mathcal{S}} \pi_i c_i < 0$, where $\boldsymbol{\pi} = [\pi_i]_{i \in \mathcal{S}}$ is the stationary distribution of the CTMC $\{J(t) : t \geq 0\}$, which implies that the process $\{(X(t), J(t)) : t \geq 0\}$ is stable. Also, by $\mathbf{0}$, $\mathbf{1}$, $\mathbf{O}$ and $\mathbf{I}$, we denote a vector of zeros, a vector of ones, a zero matrix, and an identity matrix, respectively, of suitable sizes as indicated. We denote by $1\{\cdot\}$ an indicator function.

2. MAMs for the Analysis of the SFMs

We build our methodology for the analysis for the SFMs with jumps by generalizing the following ideas and the results for the analysis of the standard SFMs. In our analysis, we apply fluid generators [5, 8, 9, 27], together with conditioning on the evolution of the sample paths of the SFM, level-crossing arguments, and corresponding physical interpretations.

Consider SFM $\{(X(t), J(t)) : t \geq 0\}$ with level variable $X(t) \geq 0$, phase variable $J(t) \in \mathcal{S}$, generator $\mathbf{T} = [T_{ij}]_{i,j \in \mathcal{S}}$, and real-valued rates $c_{J(t)} \in \mathbb{R}$. In this standard SFM, we have $dX(t)/dt = c_{J(t)}$ whenever $X(t) > 0$, and $dX(t)/dt = max\{0, c_{J(t)}\}$ whenever $X(t) = 0$. We partition the quantities below according to $\mathcal{S} = \mathcal{S}_+ \cup \mathcal{S}_- \cup \mathcal{S}_0$, in the same manner as in the SFM with jumps defined in the Introduction.

The expression for the stationary distribution of the standard SFM, as stated by O'Reilly and Scheinhardt in [25, Equations (50)-(52)], is summarised in Lemma 1 below. The result follows by conditioning on the time the process hits level zero from above, in a manner similar to Bean and O'Reilly in [6, Theorem 2], which considered a more general class of SFMs. The analysis is written *directly* in terms of the key fluid generator $\mathbf{Q}$ introduced by Bean, O'Reilly and Taylor in [8] and is suitable for a model with any *real-valued* rates, and so in a slightly different manner than an earlier form derived by Da Silva Soares in [12], which treated nonzero rates case separately (and required converting the results to the general case).

In relation to this we also make a comment in Remark 1 below.

Define matrices $\mathbf{K}(s)$, $\mathbf{K} = \mathbf{K}(0)$, $\mathbf{J}(s)$, $\mathbf{J} = \mathbf{J}(0)$, $\mathbf{D}(s)$, and $\mathbf{D} = \mathbf{D}(0)$, such that

$$\begin{aligned} \mathbf{K}(s) &= \mathbf{Q}_{++}(s) + \mathbf{\Psi}(s)\mathbf{Q}_{-+}(s), & (6)\\ \mathbf{D}(s) &= \mathbf{Q}_{--}(s) + \mathbf{Q}_{-+}(s)\mathbf{\Psi}(s), & (7)\\ \mathbf{J}(s) &= \mathbf{Q}_{--}(s) + \mathbf{\Xi}(s)\mathbf{Q}_{+-}(s), & (8) \end{aligned}$$

where $\mathbf{\Psi} = \mathbf{\Psi}(0) = [\Psi_{ij}]_{i\in\mathcal{S}_+, j\in\mathcal{S}_+}$ is the minimum nonnegative solution of an appropriate Riccati equation in [8] that has the interpretation as the probability matrix recording probabilities Ψ_{ij} of first returning to the original level $z \geq 0$ and doing so in phase $j \in \mathcal{S}_-$, given start in z in phase $i \in \mathcal{S}_+$, and $\mathbf{\Xi} = \mathbf{\Xi}(0) = [\Xi_{ij}]_{i\in\mathcal{S}_-, j\in\mathcal{S}_+}$ has a symmetric interpretation in a process without a lower boundary at 0 such that $X(t) \in (-\infty, +\infty)$ and $dX(t)/dt = c_{J(t)}$ for all t. Matrices $\mathbf{\Psi}(s)$ and $\mathbf{\Xi}(s)$ record the corresponding LSTs of the time to do so and can be computed using iterative algorithms (e.g. see [9] and references within).

Further, we let

$$\begin{aligned} \mathbf{K}^{(R)}(s) &= \mathbf{W}_{++}(s) + \mathbf{\Psi}^{(R)}(s)\mathbf{W}_{-+}(s), & (9)\\ \mathbf{J}^{(R)}(s) &= \mathbf{W}_{--}(s) + \mathbf{\Xi}^{(R)}(s)\mathbf{W}_{-+}(s), & (10)\\ \mathbf{D}^{(R)}(s) &= \mathbf{W}_{--}(s) + \mathbf{W}_{-+}(s)\mathbf{\Psi}^{(R)}(s), & (11)\\ \mathbf{U}^{(R)}(s) &= \mathbf{W}_{++}(s) + \mathbf{W}_{+-}(s)\mathbf{\Xi}^{(R)}(s), & (12) \end{aligned}$$

where $\mathbf{\Psi}^{(R)}(s)$ and $\mathbf{\Xi}^{(R)}(s)$ record the LSTs of the total reward/costs accumulated during sample paths contributing to $\mathbf{\Psi}$ and $\mathbf{\Xi}$ respectively, and note that $\mathbf{K}^{(R)}(s) = \mathbf{K}(s)$, $\mathbf{J}^{(R)}(s) = \mathbf{J}(s)$, $\mathbf{D}^{(R)}(s) = \mathbf{D}(s)$, and $\mathbf{U}^{(R)}(s) = \mathbf{U}(s)$, whenever $r_i = 1$ for all $i \in \mathcal{S}$.

2.1. Stationary analysis of standard SFMs

Define $\mathbf{P}(t,x) = [P_j(t,x)]_{j\in\mathcal{S}}$, $\mathbf{f}(t,x) = [f_j(t,x)]_{j\in\mathcal{S}}$, such that

$$\begin{aligned} P_j(t,x) &= \mathbb{P}(X(t) \leq x, J(t) = j),\\ f_j(t,x) &= \frac{\partial P_j(t,x)}{\partial x}, \end{aligned}$$

for $t > 0$, $x > 0$, $j \in \mathcal{S}$, and let $\mathbf{p} = \lim_{t\to\infty} \mathbf{P}(t,0)$ and $\boldsymbol{\pi}(x) = \lim_{t\to\infty} \mathbf{f}(t,x)$. Further, we let

$$\mathbf{F}(x) = [\mathbb{P}(X > x, J = i)]_{i\in\mathcal{S}} = [\lim_{t\to\infty} \mathbb{P}(X(t) > x, J(t) = i)]_{i\in\mathcal{S}},$$

and partition $\mathbf{p}$, $\boldsymbol{\pi}(x)$ and $\mathbf{F}(x)$ according to $\mathcal{S}_+ \cup \mathcal{S}_- \cup \mathcal{S}_0$, with

$$\mathbf{p} = \begin{bmatrix}\mathbf{0}_+ & \mathbf{p}_- & \mathbf{p}_0\end{bmatrix}, \quad \boldsymbol{\pi}(x) = \begin{bmatrix}\boldsymbol{\pi}(x)_+ & \boldsymbol{\pi}(x)_- & \boldsymbol{\pi}(x)_0\end{bmatrix}, \quad \mathbf{F}(x) = \begin{bmatrix}\mathbf{F}_+(x) & \mathbf{F}_-(x) & \mathbf{F}_0(x)\end{bmatrix},$$

respectively, and let $\mathbf{p}_\ominus = \begin{bmatrix} \mathbf{p}_- & \mathbf{p}_0 \end{bmatrix}$.

Denote

$$\mathbf{T}_{\ominus\ominus} = \begin{bmatrix} \mathbf{T}_{--} & \mathbf{T}_{-0} \\ \mathbf{T}_{0-} & \mathbf{T}_{00} \end{bmatrix}, \quad \mathbf{T}_{\ominus+} = \begin{bmatrix} \mathbf{T}_{-+} \\ \mathbf{T}_{0+} \end{bmatrix}, \quad \mathbf{T}_{\pm 0} = \begin{bmatrix} \mathbf{T}_{+0} \\ \mathbf{T}_{-0} \end{bmatrix}.$$

Lemma 1. *The vector* $\mathbf{p}_\ominus = \begin{bmatrix} \mathbf{p}_- & \mathbf{p}_0 \end{bmatrix}$ *is given by*

$$\begin{bmatrix} \mathbf{p}_- & \mathbf{p}_0 \end{bmatrix} = \alpha \begin{bmatrix} \boldsymbol{\xi}_- & \mathbf{0}_0 \end{bmatrix} (-\mathbf{T}_{\ominus\ominus})^{-1},$$

where $\boldsymbol{\xi}\_$ *is the unique solution of*

$$\begin{aligned} \boldsymbol{\xi}_- \begin{bmatrix} \mathbf{I}_- & \mathbf{O}_{-0} \end{bmatrix} (-\mathbf{T}_{\ominus\ominus})^{-1} \mathbf{T}_{\ominus+} \boldsymbol{\Psi} &= \boldsymbol{\xi}_-, \\ \boldsymbol{\xi}\_\mathbf{1} &= 1, \end{aligned}$$

and α *is a normalizing constant given by*

$$\begin{aligned} \alpha = & \Bigg\{ \begin{bmatrix} \boldsymbol{\xi}_- & \mathbf{0}_0 \end{bmatrix} (-\mathbf{T}_{\ominus\ominus})^{-1} \Bigg(\mathbf{1} + \mathbf{T}_{\ominus+} (-\mathbf{K})^{-1} \begin{bmatrix} (\mathbf{C}_+)^{-1} & \boldsymbol{\Psi}(|\mathbf{C}_-|)^{-1} \end{bmatrix} \\ & \left(\mathbf{1} + \mathbf{T}_{\pm 0} (-\mathbf{T}_{00})^{-1} \mathbf{1} \right) \Bigg) \Bigg\}^{-1}. \end{aligned}$$

For $x > 0$, $\boldsymbol{\pi}_+(x) = \begin{bmatrix} \boldsymbol{\pi}_+(x) & \boldsymbol{\pi}_-(x) & \boldsymbol{\pi}_0(x) \end{bmatrix}$ *are given by,*

$$\begin{aligned} \begin{bmatrix} \boldsymbol{\pi}_+(x) & \boldsymbol{\pi}_-(x) \end{bmatrix} &= \begin{bmatrix} \mathbf{p}_- & \mathbf{p}_0 \end{bmatrix} \mathbf{T}_{\ominus+} e^{\mathbf{K}x} \begin{bmatrix} (\mathbf{C}_+)^{-1} & \boldsymbol{\Psi}(|\mathbf{C}_-|)^{-1} \end{bmatrix}, \\ \boldsymbol{\pi}_0(x) &= \begin{bmatrix} \boldsymbol{\pi}_+(x) & \boldsymbol{\pi}_-(x) \end{bmatrix} \mathbf{T}_{\pm 0} (-\mathbf{T}_{00})^{-1}. \end{aligned}$$

Corollary 1. *For all* $x \geq 0$*, vectors* $\mathbf{F}(x) = \begin{bmatrix} \mathbf{F}_+(x) & \mathbf{F}_-(x) & \mathbf{F}_0(x) \end{bmatrix}$ *are given by,*

$$\begin{aligned} \begin{bmatrix} \mathbf{F}_+(x) & \mathbf{F}_-(x) \end{bmatrix} &= \begin{bmatrix} \mathbf{p}_- & \mathbf{p}_0 \end{bmatrix} \mathbf{T}_{\ominus+} (-\mathbf{K}^{-1}) e^{\mathbf{K}x} \begin{bmatrix} (\mathbf{C}_+)^{-1} & \boldsymbol{\Psi}(|\mathbf{C}_-|)^{-1} \end{bmatrix}, \\ \mathbf{F}_0(x) &= \begin{bmatrix} \mathbf{F}_+(x) & \mathbf{F}_-(x) \end{bmatrix} \mathbf{T}_{\pm 0} (-\mathbf{T}_{00})^{-1}. \end{aligned}$$

Proof. The result follows immediately, since $\int_{u=x}^{\infty} e^{\mathbf{K}u} du = \mathbf{K}^{-1} \left[e^{\mathbf{K}u} \right]_{u=x}^{\infty} = (-\mathbf{K}^{-1}) e^{\mathbf{K}x}$.

Remark 1. Nabli et al. have stated in [22] that "*A numerical analysis shows that MAM may be inaccurate*". The authors have supported their claim with an example in [22, Figure 5] in which they stated that they applied the results for the stationary distribution of the SFMs derived by Da Silva Soares in [12], obtaining probabilities $1-P(X>x) = 1-\sum_{i\in\mathcal{S}}\mathbb{P}(X>x, J=i) = 1-\sum_{i\in\mathcal{S}}\lim_{t\to\infty}\mathbb{P}(X(t)>x, J(t)=i)$ that exceeded 1. We note that applying the above Lemma 1 to the example in [22, Figure 5], does not result in an instability and produces accurate results, see Figure 2 below. We emphasise that MAM is a theory, which does not have an issue with accuracy.

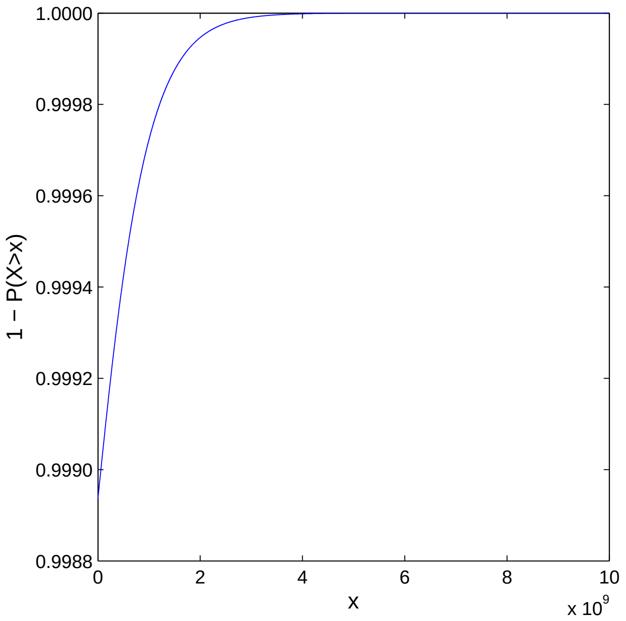

Figure 2. Application of Lemma 1 to the example in Nabli et al. [22, Figure 5] is stable.

Remark 2. Consider vectors $\boldsymbol{\pi}_-(x)$ and $\boldsymbol{\pi}_+(x)$, $x>0$. We apply analytical arguments analogous to da Silva Soares and Latouche in [13], to highlight some useful physical intepretations. By a decomposition of the sample path in which we condition on the time the process leaves level zero and does so in some phase in $\mathcal{S}_+$, we have, for $j\in\mathcal{S}_+\cup\mathcal{S}_-$, $x>0$, and all $t>0$,

$$\begin{aligned} f_j(t,x)h = &\sum_{i\in\mathcal{S}_+}\int_{\tau=0}^{t} f_i(t-\tau,0)c_i d\tau \\ &\times\mathbb{P}(X(\tau)\in(x,x+h), J(\tau)=j, X(s)>0\;\forall s\in(0,\tau]\mid X(0)=0, J(0)=i)+o(h) \\ = &\sum_{i\in\mathcal{S}_+}\int_{\tau=0}^{t} f_i(t-\tau,0)c_i d\tau \\ &\times\mathbb{P}\left(x\in X^{-1}\left(\tau,\tau+\frac{h}{|c_j|}\right), J(\tau)=j, X(s)>0\;\forall s\in(0,\tau]\mid X(0)=0, J(0)=i\right)+o(h), \end{aligned}$$

and so by dividing above by h and taking limits, we obtain

$$\pi_j(x) \;=\; \lim_{t\to\infty} f_j(t;x) = \sum_{i\in\mathcal{S}_+} \pi_i(0)c_i \int_{\tau=0}^{\infty} \left[\boldsymbol{\phi}^{(x)}(\tau)\right]_{ij} \frac{1}{|c_j|} d\tau,$$

where

$$\left[\boldsymbol{\phi}^{(x)}(t)\right]_{ij} \;=\; \frac{\partial}{\partial x}\mathbb{P}(X(t)\le x, J(t)=j, X(u)>0\;\forall u\in(0,t] \mid X(0)=0, J(0)=i)$$

is the conditional density of visiting state (x, j) at time t, avoiding level zero, given start in state $(0, i)$ at time 0. Then, by Ramaswami [26], the entry $[e^{\mathbf{K}x}]_{ij} = \int_{\tau=0}^{\infty} \left[\boldsymbol{\phi}^{(x)}(\tau)\right]_{ij} d\tau$ records the expected number of such visits over an infinite horizon, and $[e^{\mathbf{K}(s)x}]_{ij} = \int_{\tau=0}^{\infty} e^{-s\tau} \left[\boldsymbol{\phi}^{(x)}(\tau)\right]_{ij} d\tau$ is the corresponding LST of the time at which these visits occur. Therefore, by the definition of $\mathbf{K}^{(R)}(s)$ in (9) and the physical interpretation of the fluid generator $\mathbf{W}(s)$ described in the Introduction,

$$[e^{\mathbf{K}^{(R)}(s)x}]_{ij} \;=\; \int_{\tau=0}^{\infty} e^{-sR(\tau)} \left[\boldsymbol{\phi}^{(x)}(\tau)\right]_{ij} d\tau \tag{13}$$

is the corresponding LST of the total reward/cost accumulated when these visits occur.

Consider the LST matrix $\widetilde{\mathbf{P}}(\lambda)$ defined by,

$$\widetilde{\mathbf{P}}(\lambda) \;=\; \begin{bmatrix} \mathbf{0}_+ & \mathbf{p}_- & \mathbf{p}_0 \end{bmatrix} + \int_{x=0}^{\infty} e^{-\lambda x} \begin{bmatrix} \boldsymbol{\pi}_+(x) & \boldsymbol{\pi}_-(x) & \boldsymbol{\pi}_0(x) \end{bmatrix} dx,$$

and denote

$$\mathbf{T}_{\bullet+} \;=\; \begin{bmatrix} \mathbf{T}_{++} \\ \mathbf{T}_{-+} \\ \mathbf{T}_{0+} \end{bmatrix},$$

$$\mathbf{A}_{\bullet} \;=\; \begin{bmatrix} (\mathbf{C}_+)^{-1} & \boldsymbol{\Psi}(|\mathbf{C}_-|)^{-1} & ((\mathbf{C}_+)^{-1}\mathbf{T}_{+0} + \boldsymbol{\Psi}(|\mathbf{C}_-|)^{-1}\mathbf{T}_{-0})\,(-\mathbf{T}_{00})^{-1} \end{bmatrix}.$$

Corollary 2. *The Laplace-Stieltjes matrix $\widetilde{\mathbf{P}}(\lambda)$ is given by,*

$$\widetilde{\mathbf{P}}(\lambda) \;=\; \begin{bmatrix} \mathbf{0}_+ & \mathbf{p}_\ominus \end{bmatrix} \left(\mathbf{I} - \mathbf{T}_{\bullet+}(\mathbf{K}-\lambda\mathbf{I})^{-1}\mathbf{A}_\bullet\right), \tag{14}$$

where the vector $\mathbf{p}_\ominus = \begin{bmatrix} \mathbf{p}_- & \mathbf{p}_0 \end{bmatrix}$ is the solution of the set of equations,

$$\mathbf{p}_\ominus \left(\begin{bmatrix} \mathbf{0}_{\ominus+} & \mathbf{I}_\ominus \end{bmatrix} - \mathbf{T}_{\ominus+}(\mathbf{K}-\lambda\mathbf{I})^{-1}\mathbf{A}_\bullet\right)(\mathbf{T}-\lambda\mathbf{C}) \;=\; -\lambda\mathbf{p}_\ominus \begin{bmatrix} \mathbf{0}_{\ominus+} & \mathbf{I}_\ominus \end{bmatrix}\mathbf{C}, \tag{15}$$

$$\mathbf{p}_{\ominus}\left(\left[\begin{array}{cc}\mathbf{0}_{\ominus+} & \mathbf{I}_{\ominus}\end{array}\right]-\mathbf{T}_{\ominus+}(\mathbf{K})^{-1}\mathbf{A}_{\bullet}\right)\mathbf{1} = 1, \tag{16}$$

or equivalently,

$$\mathbf{p}_{\ominus}\left(\left[\begin{array}{cc}\mathbf{0}_{\ominus+} & \mathbf{I}_{\ominus}\end{array}\right]-\mathbf{T}_{\ominus+}(\mathbf{K})^{-1}\mathbf{A}_{\bullet}\right)\mathbf{T} = \mathbf{0}_{\ominus\bullet}, \tag{17}$$

$$\mathbf{p}_{\ominus}\left(\left[\begin{array}{cc}\mathbf{0}_{\ominus+} & \mathbf{I}_{\ominus}\end{array}\right]-\mathbf{T}_{\ominus+}(\mathbf{K})^{-1}\mathbf{A}_{\bullet}\right)\mathbf{1} = 1, \tag{18}$$

or equivalently,

$$\mathbf{p}_{\ominus}\left(\left[\begin{array}{cc}\mathbf{0}_{\ominus+} & \mathbf{I}_{\ominus}\end{array}\right]-\mathbf{T}_{\ominus+}(\mathbf{K})^{-1}\mathbf{A}_{\bullet}\right) = \boldsymbol{\pi}, \tag{19}$$

$$\mathbf{p}_{\ominus}\left(\left[\begin{array}{cc}\mathbf{0}_{\ominus+} & \mathbf{I}_{\ominus}\end{array}\right]-\mathbf{T}_{\ominus+}(\mathbf{K})^{-1}\mathbf{A}_{\bullet}\right)\mathbf{1} = 1, \tag{20}$$

where $\boldsymbol{\pi}$ is the stationary distribution vector of the CTMC $\{J(t) : t \geq 0\}$.

Remark 3. The above corollary leads to an alternative, slightly more convenient way of computing the stationary distribution of the SFM, than the method in Lemma 1, in which one needed to evaluate α. Here instead, we first compute $\mathbf{p}_{\ominus}$ by solving e.g. (19)-(20), and then the remaining quanities follow, as they all are expressed in terms of $\mathbf{p}_{\ominus}$.

Proof. By Lemma 1,

$$\begin{aligned}\int_{x=0}^{\infty} e^{-\lambda x}\boldsymbol{\pi}_{+}(x)dx &= \int_{x=0}^{\infty} e^{-\lambda x}\left[\begin{array}{cc}\mathbf{p}_{-} & \mathbf{p}_{0}\end{array}\right]\left[\begin{array}{c}\mathbf{T}_{-+}\\ \mathbf{T}_{0+}\end{array}\right]e^{\mathbf{K}x}(\mathbf{C}_{+})^{-1}dx\\ &= -\left[\begin{array}{cc}\mathbf{p}_{-} & \mathbf{p}_{0}\end{array}\right]\left[\begin{array}{c}\mathbf{T}_{-+}\\ \mathbf{T}_{0+}\end{array}\right](\mathbf{K}-\lambda\mathbf{I})^{-1}(\mathbf{C}_{+})^{-1}\\ &= -\mathbf{p}_{\ominus}\mathbf{T}_{\ominus+}(\mathbf{K}-\lambda\mathbf{I})^{-1}(\mathbf{C}_{+})^{-1},\end{aligned}$$

and

$$\begin{aligned}\int_{x=0}^{\infty} e^{-\lambda x}\boldsymbol{\pi}_{-}(x)dx &= \int_{x=0}^{\infty} e^{-\lambda x}\boldsymbol{\pi}_{+}(x)dx\mathbf{C}_{+}\boldsymbol{\Psi}(|\mathbf{C}_{-}|)^{-1}\\ &= -\left[\begin{array}{cc}\mathbf{p}_{-} & \mathbf{p}_{0}\end{array}\right]\left[\begin{array}{c}\mathbf{T}_{-+}\\ \mathbf{T}_{0+}\end{array}\right](\mathbf{K}-\lambda\mathbf{I})^{-1}\boldsymbol{\Psi}(|\mathbf{C}_{-}|)^{-1}\\ &= -\mathbf{p}_{\ominus}\mathbf{T}_{\ominus+}(\mathbf{K}-\lambda\mathbf{I})^{-1}\boldsymbol{\Psi}(|\mathbf{C}_{-}|)^{-1},\end{aligned}$$

and

$$\int_{x=0}^{\infty} e^{-\lambda x}\boldsymbol{\pi}_{0}(x)dx = \int_{x=0}^{\infty} e^{-\lambda x}\left[\begin{array}{cc}\boldsymbol{\pi}_{+}(x) & \boldsymbol{\pi}_{-}(x)\end{array}\right]\left[\begin{array}{c}\mathbf{T}_{+0}\\ \mathbf{T}_{-0}\end{array}\right](-\mathbf{T}_{00})^{-1}dx$$

$$= -\left[\begin{array}{cc} \mathbf{p}_- & \mathbf{p}_0 \end{array}\right]\left[\begin{array}{c} \mathbf{T}_{-+} \\ \mathbf{T}_{0+} \end{array}\right](\mathbf{K}-\lambda\mathbf{I})^{-1}\left[\begin{array}{cc} (\mathbf{C}_+)^{-1} & \mathbf{\Psi}(|\mathbf{C}_-|)^{-1} \end{array}\right]\left[\begin{array}{c} \mathbf{T}_{+0} \\ \mathbf{T}_{-0} \end{array}\right](-\mathbf{T}_{00})^{-1}$$
$$= -\mathbf{p}_\ominus\mathbf{T}_{\ominus+}(\mathbf{K}-\lambda\mathbf{I})^{-1}\left[\begin{array}{cc} (\mathbf{C}_+)^{-1} & \mathbf{\Psi}(|\mathbf{C}_-|)^{-1} \end{array}\right]\mathbf{T}_{\pm0}(-\mathbf{T}_{00})^{-1},$$

and so,

$$\begin{aligned}\widetilde{\mathbf{P}}(\lambda) &= \mathbf{p} - \mathbf{p}\mathbf{T}_{\bullet+}(\mathbf{K}-\lambda\mathbf{I})^{-1}\left[(\mathbf{C}_+)^{-1} \quad \mathbf{\Psi}(|\mathbf{C}_-|)^{-1}\right. \\ &\qquad \left.\left((\mathbf{C}_+)^{-1}\mathbf{T}_{+0} + \mathbf{\Psi}(|\mathbf{C}_-|)^{-1}\mathbf{T}_{-0}\right)(-\mathbf{T}_{00})^{-1}\right] \\ &= \mathbf{p} - \mathbf{p}\mathbf{T}_{\bullet+}(\mathbf{K}-\lambda\mathbf{I})^{-1}\mathbf{A}_\bullet,\end{aligned}$$

which gives (14).

An alternative expression for $\widetilde{\mathbf{P}}(\lambda)$ follows by applying the Kolmogorov differential equations, for example see Kulkarni and Yan [19]. That is, for $x > 0$,

$$\frac{\partial \mathbf{P}(t,x)}{\partial t} = \mathbf{P}(t,x)\mathbf{T} - \frac{\partial \mathbf{P}(t,x)}{\partial x}\mathbf{C},$$

and so by taking limits $t \to \infty$, then multiplying both sides by $e^{-\lambda x}$ and integrating with respect to x, it follows that

$$\widetilde{\mathbf{P}}(\lambda)\left(\mathbf{T} - \lambda\mathbf{C}\right) = -\lambda\mathbf{p}\mathbf{C}. \tag{21}$$

By (14) and (21), we then have (15) and by substituting $\lambda = 0$, we obtain (17). Further, since $\widetilde{\mathbf{P}}(0)\mathbf{1} = 1$, the equation (16) follows. Finally, (19) follows due to $\boldsymbol{\pi}\mathbf{T} = \mathbf{0}$ and $\boldsymbol{\pi}\mathbf{1} = 1$.

2.2. Transient analysis of standard SFMs

Transient analysis of the SFMs was derived by Ahn and Ramaswami in [2] via a stochastic coupling technique and arguments within the QBDs. Here, we derive the transient results via a direct analysis technique in which apply the physical intepretations of the fluid generators introduced in [5, 8, 9, 27], and write slightly more general results, in terms of the total accumulated reward/cost $R(t) = \int_{u=0}^{t} r_{J(u)}du$ described in the Introduction.

Assume the process starts from level $X(0) = 0$ in some phase $J(0) = i \in \mathcal{S}_- \cup \mathcal{S}_0$ according to the initial distribution vector $\boldsymbol{\alpha}(0) = [\alpha_i(0)]_{i\in\mathcal{S}_-\cup\mathcal{S}_0}$, with $\alpha_i(0) = \mathbb{P}(J(0) = i)$.

Consider the LSTs, denoted ${}_{\boldsymbol{\alpha}(0)}\mathcal{L}^{(R)}_{\mathbf{p}}(s)$ and ${}_{\boldsymbol{\alpha}(0)}\mathcal{L}^{(R)}_{\mathbf{f}}(x;s)$ for $x > 0$, defined as

$$\begin{aligned}\left[{}_{\boldsymbol{\alpha}(0)}\mathcal{L}^{(R)}_{\mathbf{p}}(s)\right]_j &= \int_{t=0}^{\infty} {}_{\boldsymbol{\alpha}(0)}\mathbb{E}(e^{-sR(t)}1\{X(t)=0, J(t)=j\})dt, \\ \left[{}_{\boldsymbol{\alpha}(0)}\mathcal{L}^{(R)}_{\mathbf{f}}(x;s)\right]_j &= \int_{t=0}^{\infty} {}_{\boldsymbol{\alpha}(0)}\mathbb{E}(e^{-sR(t)}1\{X(t)=x, J(t)=j\})dt,\end{aligned}$$

where $\left[{}_{\boldsymbol{\alpha}(0)}\mathcal{L}^{(R)}_{\mathbf{p}}(s)\right]_j$ is the LST of the total reward/cost accumulated when observing the process at level zero in some phase in $\mathcal{S}_\ominus$, and $\left[{}_{\boldsymbol{\alpha}(0)}\mathcal{L}^{(R)}_{\mathbf{f}}(x;s)\right]_j$ is the LST of the total reward/cost accumulated when observing the process at level x in some phase in $\mathcal{S}$, given the initial distribution $\boldsymbol{\alpha}(0)$.

Consider the LSTs denoted ${}_{\boldsymbol{\alpha}(0)}\mathcal{L}_{\mathbf{p}}(s)$ and ${}_{\boldsymbol{\alpha}(0)}\mathcal{L}_{\mathbf{f}}(x;s)$ for $x > 0$, partitioned according to $\mathcal{S}_- \cup \mathcal{S}_0$ and $\mathcal{S}_+ \cup \mathcal{S}_- \cup \mathcal{S}_0$ respectively, defined as

$$
\begin{aligned}
{}_{\boldsymbol{\alpha}(0)}\mathcal{L}_{\mathbf{p}}(s) &= \int_{t=0}^{\infty} e^{-st}{}_{\boldsymbol{\alpha}(0)}\mathbf{p}(t)dt = \int_{t=0}^{\infty} e^{-st}\left[\begin{array}{cc} {}_{\boldsymbol{\alpha}(0)}\mathbf{p}(t)_- & {}_{\boldsymbol{\alpha}(0)}\mathbf{p}(t)_0 \end{array}\right]dt,\\
{}_{\boldsymbol{\alpha}(0)}\mathcal{L}_{\mathbf{f}}(x;s) &= \int_{t=0}^{\infty} e^{-st}{}_{\boldsymbol{\alpha}(0)}\mathbf{f}(t,x)dt = \int_{t=0}^{\infty} e^{-st}\left[\begin{array}{ccc} {}_{\boldsymbol{\alpha}(0)}\mathbf{f}(t,x)_+ & {}_{\boldsymbol{\alpha}(0)}\mathbf{f}(t,x)_- & {}_{\boldsymbol{\alpha}(0)}\mathbf{f}(t,x)_0 \end{array}\right]dt,
\end{aligned}
$$

where

$$
\begin{aligned}
[{}_{\boldsymbol{\alpha}(0)}\mathbf{p}(t)]_i &= \mathbb{P}(X(t)=0, J(t)=i),\\
[{}_{\boldsymbol{\alpha}(0)}\mathbf{f}(t,x)]_i &= \frac{\partial}{\partial x}\mathbb{P}(X(t)\le x, J(t)=i),
\end{aligned}
$$

and note that the quantities ${}_{\boldsymbol{\alpha}(0)}\mathbf{p}(t)$ and ${}_{\boldsymbol{\alpha}(0)}\mathbf{f}(t,x)$ can be computed by numerically inverting the LSTs ${}_{\boldsymbol{\alpha}(0)}\mathcal{L}_{\mathbf{p}}(s)$ and ${}_{\boldsymbol{\alpha}(0)}\mathcal{L}_{\mathbf{f}}(x;s)$, respectively, using standard inversion algorithms, see e.g. Abate and Whitt [1], Den Iseger [14], or Horváth et al. [18]. We note that if $r_i = 1$ for all $i \in \mathcal{S}$, then ${}_{\boldsymbol{\alpha}(0)}\mathcal{L}_{\mathbf{p}}(s) = {}_{\boldsymbol{\alpha}(0)}\mathcal{L}^{(R)}_{\mathbf{p}}(s)$ and ${}_{\boldsymbol{\alpha}(0)}\mathcal{L}_{\mathbf{f}}(x;s) = {}_{\boldsymbol{\alpha}(0)}\mathcal{L}^{(R)}_{\mathbf{f}}(x;s)$.

For any matrix $\mathbf{A}$, denote by $\chi(\mathbf{A})$, the eigenvalue with maximum real part of $\mathbf{A}$.

Lemma 2. *Assume that $\chi(s\mathbf{R}_\ominus - \mathbf{T}_{\ominus\ominus}) < 0$. We have*

$$
\begin{aligned}
{}_{\boldsymbol{\alpha}(0)}\mathcal{L}^{(R)}_{\mathbf{p}}(s) &= \left[\begin{array}{cc} {}_{\boldsymbol{\alpha}(0)}\mathcal{L}^{(R)}_{\mathbf{p}}(s)_- & {}_{\boldsymbol{\alpha}(0)}\mathcal{L}^{(R)}_{\mathbf{p}}(s)_0 \end{array}\right]\\
&= \boldsymbol{\alpha}(0)\left(s\mathbf{R}_\ominus - \mathbf{T}_{\ominus\ominus}\right)^{-1} + \boldsymbol{\alpha}(0)\left(s\mathbf{R}_\ominus - \mathbf{T}_{\ominus\ominus}\right)^{-1}\mathbf{T}_{\ominus+}\left[\begin{array}{cc} \boldsymbol{\Psi}^{(R)}(s) & \mathbf{0}_{+0} \end{array}\right]\\
&\quad \times\left(\mathbf{I} - \left(s\mathbf{R}_\ominus - \mathbf{T}_{\ominus\ominus}\right)^{-1}\mathbf{T}_{\ominus+}\left[\begin{array}{cc} \boldsymbol{\Psi}^{(R)}(s) & \mathbf{0}_{+0} \end{array}\right]\right)^{-1}\left(s\mathbf{R}_\ominus - \mathbf{T}_{\ominus\ominus}\right)^{-1}
\end{aligned} \tag{22}
$$

and for $x > 0$,

$$
{}_{\boldsymbol{\alpha}(0)}\mathcal{L}^{(R)}_{\mathbf{f}}(x;s) = \left[\begin{array}{ccc} {}_{\boldsymbol{\alpha}(0)}\mathcal{L}^{(R)}_{\mathbf{f}}(x;s)_+ & {}_{\boldsymbol{\alpha}(0)}\mathcal{L}^{(R)}_{\mathbf{f}}(x;s)_- & {}_{\boldsymbol{\alpha}(0)}\mathcal{L}^{(R)}_{\mathbf{f}}(x;s)_0 \end{array}\right] \tag{23}
$$

with

$$
\begin{aligned}
&\left[\begin{array}{cc} {}_{\boldsymbol{\alpha}(0)}\mathcal{L}^{(R)}_{\mathbf{f}}(x;s)_+ & {}_{\boldsymbol{\alpha}(0)}\mathcal{L}^{(R)}_{\mathbf{f}}(x;s)_- \end{array}\right]\\
&= {}_{\boldsymbol{\alpha}(0)}\mathcal{L}^{(R)}_{\mathbf{p}}(s)\mathbf{T}_{\ominus+}e^{\mathbf{K}^{(R)}(s)x}\left[\begin{array}{cc} (\mathbf{C}_+)^{-1} & \boldsymbol{\Psi}^{(R)}(s)(|\mathbf{C}_-|)^{-1} \end{array}\right], \qquad (24)\\
&{}_{\boldsymbol{\alpha}(0)}\mathcal{L}^{(R)}_{\mathbf{f}}(x;s)_0\\
&= \left[\begin{array}{cc} {}_{\boldsymbol{\alpha}(0)}\mathcal{L}^{(R)}_{\mathbf{f}}(x;s)_+ & {}_{\boldsymbol{\alpha}(0)}\mathcal{L}^{(R)}_{\mathbf{f}}(x;s)_- \end{array}\right]\mathbf{T}_{\pm0}(s\mathbf{R}_0 - \mathbf{T}_{00})^{-1}. \qquad (25)
\end{aligned}
$$

Proof. By arguments analogous to [5, Theorem 8], since $\chi\left(s\mathbf{R}_{\ominus}-\mathbf{T}_{\ominus\ominus}\right)<0$, the inverse $\left(s\mathbf{R}_{\ominus}-\mathbf{T}_{\ominus\ominus}\right)^{-1}=\int_{t=0}^{\infty}e^{(\mathbf{T}_{\ominus\ominus}-s\mathbf{R}_{\ominus})t}dt$ exists , and its physical interpretation is that

$$\left[\left(s\mathbf{R}_{\ominus}-\mathbf{T}_{\ominus\ominus}\right)^{-1}\right]_{ij}=\int_{t=0}^{\infty}\mathbb{E}(e^{-sR(t)}1\{J(t)=j\}\mid X(0)=0,J(0)=i,\forall u\leq t,J(u)\in\mathcal{S}_{\ominus})dt$$

is the LST of the reward/cost accumulated during the time the process spends at level zero and does so in phases in $\mathcal{S}_{\ominus}$, before leaving level zero.

Expression (22) follows since for the process to be observed at level zero in some phase in $\mathcal{S}_{\ominus}$,

- it may remain at level zero without leaving it according to $\boldsymbol{\alpha}(0)\left(s\mathbf{R}_{\ominus}-\mathbf{T}_{\ominus\ominus}\right)^{-1}$,
- or it may leave level zero according to $\boldsymbol{\alpha}(0)\left(s\mathbf{R}_{\ominus}-\mathbf{T}_{\ominus\ominus}\right)^{-1}\mathbf{T}_{\ominus+}$, then come back according to $\left[\begin{array}{cc}\boldsymbol{\Psi}^{(R)}(s) & \mathbf{0}_{+0}\end{array}\right]$, and then possibly leave and come back again any number of times including zero according to $\left(\mathbf{I}-\left(s\mathbf{R}_{\ominus}-\mathbf{T}_{\ominus\ominus}\right)^{-1}\mathbf{T}_{\ominus+}\left[\begin{array}{cc}\boldsymbol{\Psi}^{(R)}(s) & \mathbf{0}_{+0}\end{array}\right]\right)^{-1}$, and then finally remain at level zero without leaving it according to $\boldsymbol{\alpha}(0)(s\mathbf{R}_{\ominus}-\mathbf{T}_{\ominus\ominus})^{-1}$.

Expression (24) follows since for the process to be observed at level x in some phase in $\mathcal{S}_{\pm}$,

- it must leave level zero for the last time, according to ${}_{\boldsymbol{\alpha}(0)}\mathcal{L}^{(R)}_{\mathbf{p}}(s)\mathbf{T}_{\ominus+}$, and then
- visit level x any number of times, while avoiding level zero, according to $e^{\mathbf{K}^{(R)}(s)x}(\mathbf{C}_{+})^{-1}$ if ending in phases in $\mathcal{S}_{+}$, or according to $e^{\mathbf{K}^{(R)}(s)x}\boldsymbol{\Psi}^{(R)}(s)(|\mathbf{C}_{-}|)^{-1}$ if ending in phases in $\mathcal{S}_{-}$.

Expression (25) follows since for the process to be observed at level x in some phase in $\mathcal{S}_{0}$,

- it must leave set $\mathcal{S}_{\pm}$ for the last time, according to $\left[\begin{array}{cc}{}_{\boldsymbol{\alpha}(0)}\mathcal{L}^{(R)}_{\mathbf{f}}(x;s)_{+} & {}_{\boldsymbol{\alpha}(0)}\mathcal{L}^{(R)}_{\mathbf{f}}(x;s)_{-}\end{array}\right]\mathbf{T}_{\pm0}$, and then
- spend some time at level x, according to $(s\mathbf{R}_{0}-\mathbf{T}_{00})^{-1}$.

Corollary 3. *We have*

$$\begin{aligned}{}_{\boldsymbol{\alpha}(0)}\mathcal{L}_{\mathbf{p}}(s) &= \left[\begin{array}{cc}{}_{\boldsymbol{\alpha}(0)}\mathcal{L}_{\mathbf{p}}(s)_{-} & {}_{\boldsymbol{\alpha}(0)}\mathcal{L}_{\mathbf{p}}(s)_{0}\end{array}\right]\\ &= \boldsymbol{\alpha}(0)\left(s\mathbf{I}-\mathbf{T}_{\ominus\ominus}\right)^{-1}+\boldsymbol{\alpha}(0)\left(s\mathbf{I}-\mathbf{T}_{\ominus\ominus}\right)^{-1}\mathbf{T}_{\ominus+}\left[\begin{array}{cc}\boldsymbol{\Psi}(s) & \mathbf{0}_{+0}\end{array}\right]\\ &\quad\times\left(\mathbf{I}-\left(s\mathbf{I}-\mathbf{T}_{\ominus\ominus}\right)^{-1}\mathbf{T}_{\ominus+}\left[\begin{array}{cc}\boldsymbol{\Psi}(s) & \mathbf{0}_{+0}\end{array}\right]\right)^{-1}\left(s\mathbf{I}-\mathbf{T}_{\ominus\ominus}\right)^{-1}, \qquad (26)\end{aligned}$$

and for $x>0$,

$${}_{\boldsymbol{\alpha}(0)}\mathcal{L}_{\mathbf{f}}(x;s) = \left[\begin{array}{ccc}{}_{\boldsymbol{\alpha}(0)}\mathcal{L}_{\mathbf{f}}(x;s)_{+} & {}_{\boldsymbol{\alpha}(0)}\mathcal{L}_{\mathbf{f}}(x;s)_{-} & {}_{\boldsymbol{\alpha}(0)}\mathcal{L}_{\mathbf{f}}(x;s)_{0}\end{array}\right] \qquad (27)$$

with

$$\begin{aligned}
&\left[\begin{array}{cc} {}_{\boldsymbol{\alpha}(0)}\mathcal{L}_{\mathbf{f}}(x;s)_{+} & {}_{\boldsymbol{\alpha}(0)}\mathcal{L}_{\mathbf{f}}(x;s)_{-} \end{array}\right] \\
= \quad & {}_{\boldsymbol{\alpha}(0)}\mathcal{L}_{\mathbf{p}}(s)\mathbf{T}_{\ominus+}e^{\mathbf{K}(s)x}\left[\begin{array}{cc} (\mathbf{C}_{+})^{-1} & \boldsymbol{\Psi}(s)(|\mathbf{C}_{-}|)^{-1} \end{array}\right], \qquad (28) \\
& {}_{\boldsymbol{\alpha}(0)}\mathcal{L}_{\mathbf{f}}(x;s)_{0} \\
= \quad & \left[\begin{array}{cc} {}_{\boldsymbol{\alpha}(0)}\mathcal{L}_{\mathbf{f}}(x;s)_{+} & {}_{\boldsymbol{\alpha}(0)}\mathcal{L}_{\mathbf{f}}(x;s)_{-} \end{array}\right]\mathbf{T}_{\pm 0}(s\mathbf{I}-\mathbf{T}_{00})^{-1}. \qquad (29)
\end{aligned}$$

Proof. The result follows from Lemma 2 by letting $\mathbf{R} = \mathbf{I}$.

3. Stationary Analysis of SFMs with Upward Jumps and Phase Transitions

Consider the stochastic fluid model (SFM) $\{(X(t), J(t)) : t \geq 0\}$ with thresholds $q_N > \ldots > q_1 > 0$, $q_0 = 0$, level variable $X(t) \geq 0$, phase variable $J(t) \in \mathcal{S}$, generator $\mathbf{T}$, and real rates $c_{J(t)} \in \mathbb{R}$, as defined in the Introduction.

Recall that we assume that the transitions at the time the process hits level $X(t) = 0$ occur according to probability matrices $\mathbf{P}^{(n)}$, $n = 0, 1 \ldots, N$.

- So, whenever a transition occurs to some $j \in \mathcal{S}_- \cup \mathcal{S}_0$ with probability according to $\mathbf{P}^{(q_0)}_{-\ominus} = [P^{(q_0)}_{ij}]_{i\in\mathcal{S}_-, j\in\mathcal{S}_\ominus}$, the process remains at the level $q_0 = 0$ for some time until a transition to some $k \in \mathcal{S}_+$ occurs at a rate according to $\mathbf{T}_{\ominus+} = [T_{ij}]_{i\in\mathcal{S}_\ominus, j\in\mathcal{S}_+}$.
- However, if a transition occurs to some $j \in \mathcal{S}_+$ with probability according to $\mathbf{P}^{(q_0)}_{-+} = [P^{(q_0)}_{ij}]_{i\in\mathcal{S}_-, j\in\mathcal{S}_+}$, the process reflects from the boundary $q_0 = 0$ and the level begins to increase.
- Similarly, whenever a transition occurs to some $j \in \mathcal{S}_0$ and level $q_n > 0$ according to $\mathbf{P}^{(n)}_{-0} = [P^{(q_0)}_{ij}]_{i\in\mathcal{S}_-, j\in\mathcal{S}_0}$, $n = 1, \ldots, N$, the process jumps to level q_n and then remains at level q_n for some time until a transition to some $k \in \mathcal{S}_+ \cup \mathcal{S}_-$ occurs at a rate according to $\mathbf{T}_{0\pm} = [T_{ij}]_{i\in\mathcal{S}_0, j\in\mathcal{S}_+\cup\mathcal{S}_-}$.
- If a transition occurs to some $j \in \mathcal{S}_- \cup \mathcal{S}_+$ and level $q_n > 0$ according to $\mathbf{P}^{(q_n)}_{-\pm} = [P^{(q_n)}_{ij}]_{i\in\mathcal{S}_0, j\in\mathcal{S}_+\cup\mathcal{S}_-}$, $n = 1, \ldots, N$, the process jumps to level q_n and instantenously leaves it, with the level increasing if the transition was to $j \in \mathcal{S}_+$, or decreasing if the transition was to $j \in \mathcal{S}_-$.

Next, we are interested in the derivation of the stationary and transient probabilities of observing levels q_n, $n = 0, 1, \ldots, N$, and the stationary and transient probability densities of observing levels $x > 0$.

For notational convenience, we introduce the following quantities, which will be used in our analysis. Let $\mathbf{P}_{--}^{(q_n)} = [P_{ij}^{(q_n)}]_{i,j\in\mathcal{S}_-}$, $\mathbf{P}_{-0}^{(q_n)} = [P_{ij}^{(q_n)}]_{i\in\mathcal{S}_-,j\in\mathcal{S}_0}$, $\mathbf{P}_{-\ominus}^{(q_n)} = [P_{ij}^{(q_n)}]_{i\in\mathcal{S}_-,j\in\mathcal{S}_\ominus}$, and $\mathbf{P}_{-+}^{(q_n)} = [P_{ij}^{(q_n)}]_{i\in\mathcal{S}_-,j\in\mathcal{S}_+}$. Define $\bar{\mathbf{P}}_{--}^{(q_n)}(s) = [\bar{P}_{ij}^{(q_n)}(s)]_{i,j\in\mathcal{S}_-}$ and $\bar{\mathbf{P}}_{-+}^{(q_n)}(s) = [\bar{P}_{ij}^{(q_n)}(s)]_{i\in\mathcal{S}_-,j\in\mathcal{S}_+}$, such that

$$\bar{\mathbf{P}}_{--}^{(q_n)}(s) = \mathbf{P}_{--}^{(q_n)} + \mathbf{P}_{-0}^{(q_n)}(s\mathbf{R}_0 - \mathbf{T}_{00})^{-1}\mathbf{T}_{0-}, \quad (30)$$

$$\bar{\mathbf{P}}_{-+}^{(q_n)}(s) = \mathbf{P}_{-+}^{(q_n)} + \mathbf{P}_{-0}^{(q_n)}(s\mathbf{R}_0 - \mathbf{T}_{00})^{-1}\mathbf{T}_{0+}, \quad (31)$$

and denote $\bar{\mathbf{P}}_{--}^{(q_n)} = \bar{\mathbf{P}}_{--}^{(q_n)}(0)$ and $\bar{\mathbf{P}}_{-+}^{(q_n)} = \bar{\mathbf{P}}_{-+}^{(q_n)}(0)$. Also, let

$$\bar{\mathbf{P}}_{-+}^{(q_0)} = \mathbf{P}_{-+}^{(q_0)} + \mathbf{P}_{-\ominus}^{(q_0)}(-\mathbf{T}_{\ominus\ominus})^{-1}\mathbf{T}_{\ominus+} \quad (32)$$

and $\mathbf{C}_\pm = \mathrm{diag}(c_i)_{i\in\mathcal{S}_+\cup\mathcal{S}_-} = \mathrm{diag}(\mathbf{C}_+, \mathbf{C}_-)$. Here, $(s\mathbf{R}_0 - \mathbf{T}_{00})^{-1} = \int_{t=0}^{\infty} e^{(\mathbf{T}_{00}-s\mathbf{R}_0)t}dt$ is the LST of the total reward/cost accumulated during the time spent in the set $\mathcal{S}_0$, which is accumulating at rate $r_{J(t)}$.

For $t > 0$, $x > 0$, let $\mathbf{P}(t,x) = [P_j(t,x)]_{j\in\mathcal{S}}$, $\mathbf{f}(t,x) = [f_j(t,x)]_{j\in\mathcal{S}}$ be probability vectors such that,

$$P_j(t,x) = \mathbb{P}(X(t) \le x, J(t) = j),$$

$$f_j(t,x) = \frac{\partial P_j(t,x)}{\partial x},$$

and let $\mathbf{p}^{(0)}(t) = [p_j^{(0)}(t)]_{j\in\mathcal{S}_\ominus}$ be such that

$$p_j^{(0)}(t) = \mathbb{P}(X(t) = 0, J(t) = j)$$

and for $q_n > 0$, let $\mathbf{p}^{(n)}(t) = [p_j^{(n)}(t)]_{j\in\mathcal{S}_0}$ be such that

$$p_j^{(n)}(t) = \mathbb{P}(X(t) = q_n, J(t) = j).$$

Note that if $\mathbf{P}_{-\ominus}^{(q_0)}\mathbf{1} = \mathbf{0}$, then $\mathbf{p}^{(0)}(t) = \mathbf{0}$, since then the probability at level zero may not accumulate due to the instantaneous jump to some level $q_n > 0$ at the moment of hitting 0 from above. Similarly, if $\mathbf{P}_{-0}^{(q_n)}\mathbf{1} = \mathbf{0}$, then $\mathbf{p}^{(n)}(t) = \mathbf{0}$.

Let $\boldsymbol{\pi} = [\pi_j]_{j\in\mathcal{S}}$ be the stationary distribution of the Markov chain $\{J(t) : t \ge 0\}$. Define probability vectors $\mathbf{F}(x) = [F_j(x)]_{j\in\mathcal{S}}$, $\mathbf{P}(x) = [P_j(x)]_{j\in\mathcal{S}}$, and $\boldsymbol{\pi}(x) = [\pi_j(x)]_{j\in\mathcal{S}}$, such that

$$F_j(x) = \mathbb{P}(X > x, J = j) = \lim_{t\to\infty}\mathbb{P}(X(t) > x, J(t) = j) = \pi_j - \lim_{t\to\infty} P_j(t,x),$$

$$P_j(x) = \mathbb{P}(X \le x, J = j) = \lim_{t\to\infty}\mathbb{P}(X(t) \le x, J(t) = j) = \lim_{t\to\infty} P_j(t,x),$$

$$f_j(x) = \frac{\partial}{\partial x}P_j(x) = \lim_{t\to\infty} f_j(t,x),$$

and let $\mathbf{p}^{(0)} = [p_j^{(0)}]_{j\in\mathcal{S}_\ominus} = \lim_{t\to\infty}\mathbf{p}^{(0)}(t)$ and $\mathbf{p}^{(n)} = [p_j^{(n)}]_{j\in\mathcal{S}_0} = \lim_{t\to\infty}\mathbf{p}^{(n)}(t)$ for $n = 1, \ldots, N$.

3.1. Key quantities

In this section, we summarise the expressions for some key building blocks that we use in our stationary as well as transient analysis. For any $0 < x \leq y$ let $\mathbf{H}^{(R)(x,y)}(s)$ and $\mathbf{G}^{(R)(x,y)}(s)$ be matrices such that $[\mathbf{H}^{(R)(x,y)}(s)]_{ij}$ is the LST of the distribution of the total reward/cost at the time the process hits level y and do so in phase j before a visit to level zero, given start from level x in phase i; and $[\mathbf{G}^{(R)(x,y)}(s)]_{ij}$ is the LST of the distribution of the total reward/cost at the time the process hits level zero and do so in phase j before a visit to level y, given start from level x in phase i.

The expression for $\mathbf{H}^{(R)(x,y)}(s)$ and $\mathbf{G}^{(R)(x,y)}(s)$, partitioned according to $\mathcal{S}_+ \cup \mathcal{S}_-\mathcal{S}_0$ as

$$\mathbf{G}^{(R)(x,y)}(s) = \begin{bmatrix} \mathbf{O}_{++} & \mathbf{G}^{(R)(x,y)}_{+-}(s) \\ \mathbf{O}_{-+} & \mathbf{G}^{(R)(x,y)}_{--}(s) \end{bmatrix}, \quad \mathbf{H}^{(R)(x,y)}(s) = \begin{bmatrix} \mathbf{H}^{(R)(x,y)}_{++}(s) & \mathbf{O}_{+-} \\ \mathbf{H}^{(R)(x,y)}_{-+}(s) & \mathbf{O}_{--} \end{bmatrix},$$

follow by arguments analogous to Bean, O'Reilly and Taylor [8] and the the physical interpretation of the fluid generator $\mathbf{W}(s)$ described in [5, 27], with

$$\begin{aligned} &\begin{bmatrix} \mathbf{O}_{++} & \mathbf{G}^{(R)(x,y)}_{+-}(s) & \mathbf{H}^{(R)(x,y)}_{++}(s) & \mathbf{O}_{+-} \\ \mathbf{O}_{-+} & \mathbf{G}^{(R)(x,y)}_{--}(s) & \mathbf{H}^{(R)(x,y)}_{-+}(s) & \mathbf{O}_{--} \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{G}^{(R)(x)}(s) & \mathbf{H}^{(R)(y-x)}(s) \end{bmatrix} \begin{bmatrix} \mathbf{I} & \mathbf{H}^{(R)(y)}(s) \\ \mathbf{G}^{(R)(y)}(s) & \mathbf{I} \end{bmatrix}^{-1} \end{aligned} \tag{33}$$

where

$$\begin{aligned} \mathbf{G}^{(R)(y)}(s) &= \begin{bmatrix} \mathbf{O}_{++} & \mathbf{\Psi}^{(R)}(s) e^{\left(\mathbf{W}_{--}(s)+\mathbf{W}_{-+}(s)\mathbf{\Psi}^{(R)}(s)\right)y} \\ \mathbf{O}_{-+} & e^{\left(\mathbf{W}_{--}(s)+\mathbf{W}_{-+}(s)\mathbf{\Psi}^{(R)}(s)\right)y} \end{bmatrix}, \\ \mathbf{H}^{(R)(y)}(s) &= \begin{bmatrix} e^{\left(\mathbf{W}_{++}(s)+\mathbf{W}_{+-}(s)\mathbf{\Xi}^{(R)}(s)\right)y} & \mathbf{O}_{+-} \\ \mathbf{\Xi}^{(R)}(s) e^{\left(\mathbf{W}_{++}(s)+\mathbf{W}_{+-}(s)\mathbf{\Xi}^{(R)}(s)\right)y} & \mathbf{O}_{--} \end{bmatrix}, \end{aligned}$$

and

$$\begin{aligned} &, \begin{bmatrix} \mathbf{I} & \mathbf{H}^{(R)(y)}(s) \\ \mathbf{G}^{(R)(y)}(s) & \mathbf{I} \end{bmatrix}^{-1} \\ &= \begin{bmatrix} (\mathbf{I} - \mathbf{H}^{(R)(y)}(s)\mathbf{G}^{(R)(y)}(s))^{-1} & -\mathbf{H}^{(R)(y)}(s)(\mathbf{I} - \mathbf{G}^{(R)(y)}(s)\mathbf{H}^{(R)(y)}(s))^{-1} \\ -\mathbf{G}^{(R)(y)}(s)(\mathbf{I} - \mathbf{H}^{(R)(y)}(s)\mathbf{G}^{(R)(y)}(s))^{-1} & (\mathbf{I} - \mathbf{G}^{(R)(y)}(s)\mathbf{H}^{(R)(y)}(s))^{-1} \end{bmatrix}, \end{aligned}$$

Denote $\mathbf{H}(x,y) = \mathbf{H}^{(R)(x,y)}(0)$ and $\mathbf{G}(x,y) = \mathbf{G}^{(R)(x,y)}(0)$ when $\mathbf{R} = \mathbf{I}$, partitioned according to $\mathcal{S}_+ \cup \mathcal{S}_- \mathcal{S}_0$,

$$\mathbf{G}(x,y) = \begin{bmatrix} \mathbf{O}_{++} & \mathbf{G}_{+-}(x,y) \\ \mathbf{O}_{-+} & \mathbf{G}_{--}(x,y) \end{bmatrix}, \quad \mathbf{H}(x,y) = \begin{bmatrix} \mathbf{H}_{++}(x,y) & \mathbf{O}_{+-} \\ \mathbf{H}_{-+}(x,y) & \mathbf{O}_{--} \end{bmatrix}.$$

For any $0 < x < q$ let $\mathbf{N}^{[q]}(q,x)$ and $\mathbf{N}^{[q]}(0,x)$ be matrices such that $[\mathbf{N}^{[q]}(q,x)]_{ij}$ is the expected number of visits to level x and doing so in phase j before a visit to level q or level zero, given start from level q in phase i; and $[\mathbf{N}^{[q]}(0,x)]_{ij}$ is the expected number of visits to level x and doing so in phase j before a visit to level q or level zero, given start from level 0 in phase i.

The expression for $\mathbf{N}^{[q]}(q,x)$ and $\mathbf{N}^{[q]}(0,x)$ follow by Bean, O'Reilly and Sargison [7], with

$$\begin{bmatrix} \mathbf{N}^{[q]}_{++}(0,x) & \mathbf{N}^{[q]}_{+-}(0,x) \\ \mathbf{N}^{[q]}_{-+}(q,x) & \mathbf{N}^{[q]}_{--}(q,x) \end{bmatrix} = \begin{bmatrix} \mathbf{I} & e^{\mathbf{K}q}\mathbf{\Psi} \\ e^{\mathbf{J}q}\mathbf{\Xi} & \mathbf{I} \end{bmatrix}^{-1} \begin{bmatrix} e^{\mathbf{K}x} & e^{\mathbf{K}x}\mathbf{\Psi} \\ e^{\mathbf{J}(q-x)}\mathbf{\Xi} & e^{\mathbf{J}(q-x)} \end{bmatrix},$$

where by

$$\begin{aligned} \begin{bmatrix} \mathbf{I} & \mathbf{A} \\ \mathbf{B} & \mathbf{I} \end{bmatrix}^{-1} &= \begin{bmatrix} (\mathbf{I}-\mathbf{AB})^{-1} & -\mathbf{A}(\mathbf{I}-\mathbf{BA})^{-1} \\ -\mathbf{B}(\mathbf{I}-\mathbf{AB})^{-1} & (\mathbf{I}-\mathbf{BA})^{-1} \end{bmatrix} \\ &= \begin{bmatrix} (\mathbf{I}-\mathbf{AB})^{-1} & -(\mathbf{I}-\mathbf{AB})^{-1}\mathbf{A} \\ -(\mathbf{I}-\mathbf{BA})^{-1}\mathbf{B} & (\mathbf{I}-\mathbf{BA})^{-1} \end{bmatrix} \end{aligned}$$

we have,

$$\begin{bmatrix} \mathbf{I} & e^{\mathbf{K}q}\mathbf{\Psi} \\ e^{\mathbf{J}q}\mathbf{\Xi} & \mathbf{I} \end{bmatrix}^{-1} = \begin{bmatrix} (\mathbf{I}-e^{\mathbf{K}q}\mathbf{\Psi}e^{\mathbf{J}q}\mathbf{\Xi})^{-1} & -(\mathbf{I}-e^{\mathbf{K}q}\mathbf{\Psi}e^{\mathbf{J}q}\mathbf{\Xi})^{-1}e^{\mathbf{K}q}\mathbf{\Psi} \\ -(\mathbf{I}-e^{\mathbf{J}q}\mathbf{\Xi}e^{\mathbf{K}q}\mathbf{\Psi})^{-1}e^{\mathbf{J}q}\mathbf{\Xi} & (\mathbf{I}-e^{\mathbf{J}q}\mathbf{\Xi}e^{\mathbf{K}q}\mathbf{\Psi})^{-1} \end{bmatrix},$$

which gives,

$$\begin{aligned} \mathbf{N}^{[q]}_{++}(0,x) &= (\mathbf{I}-e^{\mathbf{K}q}\mathbf{\Psi}e^{\mathbf{J}q}\mathbf{\Xi})^{-1}(e^{\mathbf{K}x}-e^{\mathbf{K}q}\mathbf{\Psi}e^{\mathbf{J}(q-x)}\mathbf{\Xi}), \\ \mathbf{N}^{[q]}_{+-}(0,x) &= (\mathbf{I}-e^{\mathbf{K}q}\mathbf{\Psi}e^{\mathbf{J}q}\mathbf{\Xi})^{-1}(e^{\mathbf{K}x}\mathbf{\Psi}-e^{\mathbf{K}q}\mathbf{\Psi}e^{\mathbf{J}(q-x)}), \\ \mathbf{N}^{[q]}_{-+}(q,x) &= (\mathbf{I}-e^{\mathbf{J}q}\mathbf{\Xi}e^{\mathbf{K}q}\mathbf{\Psi})^{-1}(-e^{\mathbf{J}q}\mathbf{\Xi}e^{\mathbf{K}x}+e^{\mathbf{J}(q-x)}\mathbf{\Xi}), \\ \mathbf{N}^{[q]}_{--}(q,x) &= (\mathbf{I}-e^{\mathbf{J}q}\mathbf{\Xi}e^{\mathbf{K}q}\mathbf{\Psi})^{-1}(-e^{\mathbf{J}q}\mathbf{\Xi}e^{\mathbf{K}x}\mathbf{\Psi}+e^{\mathbf{J}(q-x)}). \end{aligned} \tag{34}$$

3.2. Embedded Markov chain #0: conditioning on visiting level q_0

To derive the stationary distribution of the SFM $\{(X(t), J(t)) : t \geq 0\}$, it is convenient to consider a discrete-time Markov chain (labelled #0) observed at the times the process *visits* level q_0. The stationary probability of such chain records proportion of times the SFM is in various phases in $\mathcal{S}_-$ when visiting levels q_0. The key idea here is to derive the stationary distribution of such Markov chain and then use it to write the expressions for the stationary distribution of our SFM.

So, consider a discrete-time Markov chain $\{(\varphi_w^{(0)}) : w = 0, 1, \ldots\}$ with state space $\mathcal{S}_-$ observed at the times when the SFM $\{(X(t), J(t)) : t \geq 0\}$ *visits* level q_0. Let $\boldsymbol{\xi}_-^{(0)} = \left[\xi_i^{(0)}\right]_{i\in\mathcal{S}_-}$, $\xi_i^{(0)} = \lim_{w\to\infty} \mathbb{P}(\varphi_w^{(0)} = i)$, be the stationary distribution vector of this chain, which is the unique solution of

$$\begin{aligned} \boldsymbol{\xi}_-^{(0)}\mathbf{P}_{--} &= \boldsymbol{\xi}_-^{(0)}, && (35)\\ \boldsymbol{\xi}_-^{(0)}\mathbf{1} &= 1, && (36) \end{aligned}$$

where

$$\mathbf{P}_{--} = \bar{\mathbf{P}}_{-+}^{(q_0)}\boldsymbol{\Psi} + \sum_{n=1}^{N}\left(\bar{\mathbf{P}}_{--}^{(q_n)} + \bar{\mathbf{P}}_{-+}^{(q_n)}\boldsymbol{\Psi}\right)e^{\left(\mathbf{Q}_{--}+\mathbf{Q}_{-+}\boldsymbol{\Psi}\right)q_n}. \tag{37}$$

Denote $b_n = q_{n+1} - q_n$ for $n = 0, 1, \ldots, N-1$.

Theorem 1. *We have,*

$$\left[\begin{array}{cc}\mathbf{p}_-^{(0)} & \mathbf{p}_0^{(0)}\end{array}\right] = \beta\boldsymbol{\xi}_-^{(0)}\mathbf{P}_{-\ominus}^{(q_0)}\left(-\mathbf{T}_{\ominus\ominus}\right)^{-1}, \tag{38}$$

and for $n > 0$,

$$\mathbf{p}_0^{(n)} = \beta\boldsymbol{\xi}_-^{(0)}\mathbf{P}_0^{(q_n)}\left(-\mathbf{T}_{00}\right)^{-1}. \tag{39}$$

Further, for $x \in (q_n, q_{n+1})$, $n = 0, 1\ldots, N-1$, and $x > q_N$, we have

$$\begin{aligned} \boldsymbol{\pi}_+(x) = \beta\boldsymbol{\xi}_-^{(0)}\Bigg(&\bar{\mathbf{P}}_{-+}^{(q_0)}e^{\mathbf{K}x}\\ &+\sum_{q_n:0<q_n<x}\left(\bar{\mathbf{P}}_{--}^{(q_n)}\mathbf{H}_{-+}(q_n,q_n) + \bar{\mathbf{P}}_{-+}^{(q_n)}\right)\mathbf{H}_{++}(q_n,x)\left(\mathbf{I}-\boldsymbol{\Psi}\mathbf{H}_{-+}(x,x)\right)^{-1}\\ &+\sum_{q_n:q_n>x}\left(\bar{\mathbf{P}}_{--}^{(q_n)} + \bar{\mathbf{P}}_{-+}^{(q_n)}\boldsymbol{\Psi}\right)e^{\mathbf{D}(q_n-x)}\mathbf{H}_{-+}(x,x)\left(\mathbf{I}-\boldsymbol{\Psi}\mathbf{H}_{-+}(x,x)\right)^{-1}\Bigg)(\mathbf{C}_+)^{-1}, \end{aligned} \tag{40}$$

and

$$\boldsymbol{\pi}_-(x) = \beta\boldsymbol{\xi}_-^{(0)}\Bigg(\bar{\mathbf{P}}_{-+}^{(q_0)}e^{\mathbf{K}x}\boldsymbol{\Psi} + \sum_{q_n:0<q_n<x}\left(\bar{\mathbf{P}}_{--}^{(q_n)}\mathbf{H}_{-+}(q_n,q_n) + \bar{\mathbf{P}}_{-+}^{(q_n)}\right)\mathbf{H}_{++}(q_n,x)\left(\mathbf{I}-\boldsymbol{\Psi}\mathbf{H}_{-+}(x,x)\right)^{-1}\boldsymbol{\Psi} + \sum_{q_n:q_n>x}\left(\bar{\mathbf{P}}_{--}^{(q_n)} + \bar{\mathbf{P}}_{-+}^{(q_n)}\boldsymbol{\Psi}\right)e^{\mathbf{D}(q_n-x)}\left(\mathbf{I}-\mathbf{H}_{-+}(x,x)\boldsymbol{\Psi}\right)^{-1}\Bigg)(|\mathbf{C}_-|)^{-1}, \quad (41)$$

and

$$\boldsymbol{\pi}_0(x) = \begin{bmatrix} \boldsymbol{\pi}_+(x) & \boldsymbol{\pi}_-(x) \end{bmatrix}\mathbf{T}_{\pm 0}(\mathbf{I}-\mathbf{T}_{00})^{-1}. \quad (42)$$

The normalising constant β is given by,

$$\beta = \Bigg\{\boldsymbol{\xi}_-^{(0)}\mathbf{P}_{-\ominus}^{(q_0)}\left(-\mathbf{T}_{\ominus\ominus}\right)^{-1}\mathbf{1}_\ominus + \sum_{q_n:0<q_n}\boldsymbol{\xi}_-^{(0)}\mathbf{P}_0^{(q_n)}\left(-\mathbf{T}_{00}\right)^{-1}\mathbf{1}_\ominus + \boldsymbol{\xi}_-^{(q_0)}\Bigg(\bar{\mathbf{P}}_{-+}^{(q_0)}(-\mathbf{K})^{-1}\begin{bmatrix} (\mathbf{C}_+)^{-1} & \boldsymbol{\Psi}(|\mathbf{C}_-|)^{-1}\end{bmatrix} + \sum_{q_n:0<q_n}\left(\bar{\mathbf{P}}_{--}^{(q_n)}\mathbf{H}_{-+}(q_n,q_n) + \bar{\mathbf{P}}_{-+}^{(q_n)}\right)\left(\mathbf{I}-\boldsymbol{\Psi}\mathbf{H}_{-+}(q_n,q_n)\right)^{-1} (-\mathbf{K})^{-1}\begin{bmatrix} (\mathbf{C}_+)^{-1} & \boldsymbol{\Psi}(|\mathbf{C}_-|)^{-1}\end{bmatrix} + \sum_{q_n:q_n>0}\left(\bar{\mathbf{P}}_{--}^{(q_n)} + \bar{\mathbf{P}}_{-+}^{(q_n)}\boldsymbol{\Psi}\right)\left(\mathbf{I}-\mathbf{H}_{-+}(q_n,q_n)\boldsymbol{\Psi}\right)^{-1}\left(\mathbf{I}-e^{\mathbf{J}q_n}\boldsymbol{\Xi}e^{\mathbf{K}q_n}\boldsymbol{\Psi}\right)^{-1} \times \begin{bmatrix} -e^{\mathbf{J}q_n}\boldsymbol{\Xi}(\mathbf{K})^{-1}(e^{\mathbf{K}q_n}-\mathbf{I}) & \left(\int_{z=0}^{q_n}e^{\mathbf{J}z}dz\right)\end{bmatrix}\begin{bmatrix}\mathbf{I} & \boldsymbol{\Psi} \\ \boldsymbol{\Xi} & \mathbf{I}\end{bmatrix}\Bigg) \left(\mathbf{1}_\pm + \mathbf{T}_{\pm 0}(-\mathbf{T}_{00})^{-1}\mathbf{1}_0\right)\Bigg\}^{-1}. \quad (43)$$

Proof. Denote $\boldsymbol{\pi}_-(q_0^+) = \lim_{x\downarrow q_0}\boldsymbol{\pi}_-(x)$. We apply standard level-crossing arguments in the decomposition of sample paths of our SFM. First, by the uniqueness of $\boldsymbol{\xi}_-^{(0)}$ and by noticing that $\boldsymbol{\pi}_-(q_0^+)|\mathbf{C}_-|$ is a solution of (35), it follows that $\boldsymbol{\pi}_-(q_0^+)|\mathbf{C}_-| = \beta\boldsymbol{\xi}_-^{(0)}$ for some constant $\beta > 0$. Next, (38)-(39) follow, since for the process to be observed with some probability at level q_0 in some phase in $\mathcal{S}_- \cup \mathcal{S}_0$, it must visit level q_0, then transition to some phase in $\mathcal{S}_- \cup \mathcal{S}_0$, and then remain at level q_0 for some time, according to

$$\begin{bmatrix} \mathbf{p}_-^{(0)} & \mathbf{p}_0^{(0)} \end{bmatrix} = \boldsymbol{\pi}_-(q_0^+)|\mathbf{C}_-| \times \mathbf{P}_{-\ominus}^{(q_0)} \times \left(-\mathbf{T}_{\ominus\ominus}\right)^{-1} = \beta\boldsymbol{\xi}_-^{(0)}\mathbf{P}_{-\ominus}^{(q_0)}\left(-\mathbf{T}_{\ominus\ominus}\right)^{-1};$$

while for the process to be observed with some probability at level $q_n > 0$ in some phase in $\mathcal{S}_0$, it must visit level q_0, then jump to level q_n, and then remain at level q_0 for some time, according to

$$\mathbf{p}_0^{(n)} \;=\; \boldsymbol{\pi}_-(q_0^+)|\mathbf{C}_-| \times \mathbf{P}_0^{(q_n)} \times (-\mathbf{T}_{00})^{-1} = \beta\boldsymbol{\xi}_-^{(0)}\mathbf{P}_0^{(q_n)}(-\mathbf{T}_{00})^{-1}\,. \tag{44}$$

Next, (40) follows since for the process to be observed at level $x > 0$ in some phase in $\mathcal{S}_+$ with probability density $\boldsymbol{\pi}_+(x)$, it must visit level q_0, and then

- reflect from it and then visit level x in some phase in $\mathcal{S}_+$ any number of times without visiting level zero (according to the first line in (40)), or
- jump to some level $q_n < x$, and then visit level x any number of times without visiting level zero (according to the second line in (40)), or
- jump to some level $q_n > x$, and then visit level x any number of times without visiting level zero (according to the third line in (40)).

We obtain (41) by a similar conditioning .

Further, rewriting the terms in $\boldsymbol{\pi}_+(x)$ and $\boldsymbol{\pi}_-(x)$ in an equivalent form,

$$\begin{aligned}\boldsymbol{\pi}_+(x) = \beta\boldsymbol{\xi}_-^{(0)}\Bigg(&\bar{\mathbf{P}}_{-+}^{(q_0)}e^{\mathbf{K}x}\\ &+\sum_{q_n:0<q_n<x}\left(\bar{\mathbf{P}}_{--}^{(q_n)}\mathbf{H}_{-+}(q_n,q_n)+\bar{\mathbf{P}}_{-+}^{(q_n)}\right)\left(\mathbf{I}-\boldsymbol{\Psi}\mathbf{H}_{-+}(q_n,q_n)\right)^{-1}e^{\mathbf{K}(x-q_n)}\\ &+\sum_{q_n:q_n>x}\left(\bar{\mathbf{P}}_{--}^{(q_n)}+\bar{\mathbf{P}}_{-+}^{(q_n)}\boldsymbol{\Psi}\right)\left(\mathbf{I}-\mathbf{H}_{-+}(q_n,q_n)\boldsymbol{\Psi}\right)^{-1}\mathbf{N}_{-+}^{[q_n]}(q_n,x)\Bigg)(\mathbf{C}_+)^{-1},\end{aligned} \tag{45}$$

and

$$\begin{aligned}\boldsymbol{\pi}_-(x) =\beta\boldsymbol{\xi}_-^{(0)}\Bigg(&\bar{\mathbf{P}}_{-+}^{(q_0)}e^{\mathbf{K}x}\boldsymbol{\Psi}\\ &+\sum_{q_n:0<q_n<x}\left(\bar{\mathbf{P}}_{--}^{(q_n)}\mathbf{H}_{-+}(q_n,q_n)+\bar{\mathbf{P}}_{-+}^{(q_n)}\right)\left(\mathbf{I}-\boldsymbol{\Psi}\mathbf{H}_{-+}(q_n,q_n)\right)^{-1}e^{\mathbf{K}(x-q_n)}\boldsymbol{\Psi}\\ &+\sum_{q_n:q_n>x}\left(\bar{\mathbf{P}}_{--}^{(q_n)}+\bar{\mathbf{P}}_{-+}^{(q_n)}\boldsymbol{\Psi}\right)\left(\mathbf{I}-\mathbf{H}_{-+}(q_n,q_n)\boldsymbol{\Psi}\right)^{-1}\mathbf{N}_{--}^{[q_n]}(q_n,x)\Bigg)(|\mathbf{C}_-|)^{-1},\end{aligned} \tag{46}$$

and integrating

$$\int_{x=0}^{\infty}e^{\mathbf{K}x}dx = (\mathbf{K})^{-1}\left[e^{\mathbf{K}x}\right]_{x=0}^{\infty} = (-\mathbf{K})^{-1}, \tag{47}$$

$$\int_{x=q_n}^{\infty}e^{\mathbf{K}(x-q_n)}dx = \int_{x=0}^{\infty}e^{\mathbf{K}x}dx = (-\mathbf{K})^{-1}, \tag{48}$$

$$\int_{x=0}^{q_n} \mathbf{N}_{-+}^{[q_n]}(q_n, x)dx = (\mathbf{I} - e^{\mathbf{J}q_n}\boldsymbol{\Xi}e^{\mathbf{K}q_n}\boldsymbol{\Psi})^{-1}\left(-e^{\mathbf{J}q_n}\boldsymbol{\Xi}(\mathbf{K})^{-1}(e^{\mathbf{K}q_n} - \mathbf{I}) + \left(\int_{z=0}^{q_n} e^{\mathbf{J}z}dz\right)\boldsymbol{\Xi}\right), \tag{49}$$

$$\int_{x=0}^{q_n} \mathbf{N}_{--}^{[q_n]}(q_n, x)dx = (\mathbf{I} - e^{\mathbf{J}q_n}\boldsymbol{\Xi}e^{\mathbf{K}q_n}\boldsymbol{\Psi})^{-1}\left(-e^{\mathbf{J}q_n}\boldsymbol{\Xi}(\mathbf{K})^{-1}(e^{\mathbf{K}q_n} - \mathbf{I})\boldsymbol{\Psi} + \left(\int_{z=0}^{q_n} e^{\mathbf{J}z}dz\right)\right), \tag{50}$$

results in the expression for β.

Remark 4. Since $\mu < 0$, it follows by Da Silva Soares [12, Theorem 3.7.2] that the matrix $\mathbf{J}$ is singular. The integral $\int_{z=0}^{q} e^{\mathbf{J}z}dz$ can be computed using the method in [12, Lemma 3.7.3], with

$$\int_{z=0}^{q} e^{\mathbf{J}z}dz = \left[\mathbf{J}^{\#}e^{\mathbf{J}x} + x\mathbf{vu}\right]_{x=0}^{x=q} = \mathbf{J}^{\#}(e^{\mathbf{J}q} - \mathbf{I}) + q\mathbf{vu},$$

where $\mathbf{J}^{\#}$ is the solution to

$$\begin{aligned}\mathbf{J}^{\#}\mathbf{J} &= \mathbf{I} - \mathbf{vu},\\ \mathbf{J}^{\#}\mathbf{v} &= \mathbf{0},\end{aligned}$$

and $\mathbf{u}$, $\mathbf{v}$ are the left/right eigenvectors of $\mathbf{J}$ for the eigenvalue 0 normalised with $\mathbf{uv} = 1$, $\mathbf{u1} = 1$.

Below, we state the result for the model studied by Kulkarni and Yan in [19], which is a special case of the model considered here.

Corollary 4. *Suppose* $\mathbf{P}^{(q_0)}\mathbf{1} = \mathbf{0}$ *and* $N = 1$. *Denote* $q = q_N$. *Then,*

$$\mathbf{p}_0^{(1)} = \beta\boldsymbol{\xi}_0^{(q_1)}\mathbf{P}_0^{(q_1)}\left(-\mathbf{T}_{00}\right)^{-1},$$

and for $x \in (0, q_1)$ *and* $x > q_1$, *we have*

$$\begin{aligned}\boldsymbol{\pi}_{+}(x) =&\beta\boldsymbol{\xi}_{-}^{(0)}\Bigg(1\{q_1 < x\}\left(\bar{\mathbf{P}}_{--}^{(q_1)}\mathbf{H}_{-+}(q_1, q_1) + \bar{\mathbf{P}}_{-+}^{(q_1)}\right)\mathbf{H}_{++}(q_1, x)\left(\mathbf{I} - \boldsymbol{\Psi}\mathbf{H}_{-+}(x, x)\right)^{-1}\\ &+ 1\{q_1 > x\}\left(\bar{\mathbf{P}}_{--}^{(q_1)} + \bar{\mathbf{P}}_{-+}^{(q_1)}\boldsymbol{\Psi}\right)e^{\mathbf{D}(q_1 - x)}\mathbf{H}_{-+}(x, x)\left(\mathbf{I} - \boldsymbol{\Psi}\mathbf{H}_{-+}(x, x)\right)^{-1}\Bigg)(\mathbf{C}_{+})^{-1},\\ \boldsymbol{\pi}_{-}(x) =&\beta\boldsymbol{\xi}_{-}^{(0)}\Bigg(1\{q_1 < x\}\left(\bar{\mathbf{P}}_{--}^{(q_1)}\mathbf{H}_{-+}(q_1, q_1) + \bar{\mathbf{P}}_{-+}^{(q_1)}\right)\mathbf{H}_{++}(q_1, x)\left(\mathbf{I} - \boldsymbol{\Psi}\mathbf{H}_{-+}(x, x)\right)^{-1}\boldsymbol{\Psi}\\ &+ 1\{q_1 > x\}\left(\bar{\mathbf{P}}_{--}^{(q_1)} + \bar{\mathbf{P}}_{-+}^{(q_1)}\boldsymbol{\Psi}\right)e^{\mathbf{D}(q_1 - x)}\left(\mathbf{I} - \mathbf{H}_{-+}(x, x)\boldsymbol{\Psi}\right)^{-1}\Bigg)(|\mathbf{C}_{-}|)^{-1},\end{aligned}$$

$$\boldsymbol{\pi}_0(x) = \left[\begin{array}{cc} \boldsymbol{\pi}_+(x) & \boldsymbol{\pi}_-(x) \end{array} \right] \mathbf{T}_{\pm 0}(\mathbf{I} - \mathbf{T}_{00})^{-1}.$$

where

$$\beta = \Bigg\{ \boldsymbol{\xi}_-^{(0)} \mathbf{P}_0^{(q_1)} \left(-\mathbf{T}_{00}\right)^{-1} \mathbf{1}_\ominus$$
$$+ \boldsymbol{\xi}_-^{(q_0)} \Bigg(\left(\bar{\mathbf{P}}_{--}^{(q_1)} \mathbf{H}_{-+}(q_1,q_1) + \bar{\mathbf{P}}_{-+}^{(q_1)} \right) \left(\mathbf{I} - \boldsymbol{\Psi}\mathbf{H}_{-+}(q_1,q_1)\right)^{-1}$$
$$(-\mathbf{K})^{-1} \left[\begin{array}{cc} (\mathbf{C}_+)^{-1} & \boldsymbol{\Psi}(|\mathbf{C}_-|)^{-1} \end{array} \right]$$
$$+ \left(\bar{\mathbf{P}}_{--}^{(q_1)} + \bar{\mathbf{P}}_{-+}^{(q_1)} \boldsymbol{\Psi} \right) \left(\mathbf{I} - \mathbf{H}_{-+}(q_1,q_1)\boldsymbol{\Psi}\right)^{-1} \left(\mathbf{I} - e^{\mathbf{J}q_1} \boldsymbol{\Xi} e^{\mathbf{K}q_1} \boldsymbol{\Psi}\right)^{-1}$$
$$\times \left[\begin{array}{cc} -e^{\mathbf{J}q_1}\boldsymbol{\Xi}(\mathbf{K})^{-1}(e^{\mathbf{K}q_1} - \mathbf{I}) & \left(\int_{z=0}^{q_1} e^{\mathbf{J}z} dz\right) \end{array} \right] \left[\begin{array}{cc} \mathbf{I} & \boldsymbol{\Psi} \\ \boldsymbol{\Xi} & \mathbf{I} \end{array} \right] \Bigg)$$
$$\left(\mathbf{1}_\pm + \mathbf{T}_{\pm 0}(-\mathbf{T}_{00})^{-1} \mathbf{1}_0 \right) \Bigg\}^{-1}.$$

and

$$\mathbf{P}_{--} = \left(\bar{\mathbf{P}}_{--}^{(q)} + \bar{\mathbf{P}}_{-+}^{(q)} \boldsymbol{\Psi} \right) e^{\mathbf{D}q}.$$

3.3. Embedded Markov chain #1: conditioning on visiting levels q_n

We also consider a discrete-time Markov chain (labelled #1) observed at the times the process *visits* one of the levels q_n, $n = 0, 1, \ldots, N$. Such visits may occur due to jumps to levels $q_n > 0$, or due to the fluid level increasing or decreasing in some phase in $\mathcal{S}_+ \cup \mathcal{S}_-$. The stationary probability of such chain records proportion of times the SFM visits levels q_n, and does so in some phase in $\mathcal{S}_-$ when visiting level q_0, or in some phase in $\mathcal{S}_+ \cup \mathcal{S}_- \cup \mathcal{S}_0$ when visiting level q_n for $n = 1, \ldots, N$. We derive the stationary distribution of such Markov chain and then use it to write the expressions for the stationary distribution of our SFM.

So, consider a discrete-time Markov chain $\{(Y_t^{(1)}, \varphi_w^{(1)}) : w = 0, 1, \ldots\}$ with state space $(\{0\} \times \mathcal{S}_-) \cup (\{1, \ldots, N\} \times (\mathcal{S}_+ \cup \mathcal{S}_- \cup \mathcal{S}_0))$ observed at the times when the SFM $\{(X(t), J(t)) : t \geq 0\}$ *visits* one of the levels q_n, $n = 0, 1, \ldots, N$.

Let $\boldsymbol{\xi} = \left[\boldsymbol{\xi}_-^{(q_0)}, \boldsymbol{\xi}^{(q_1)}, \ldots, \boldsymbol{\xi}^{(q_N)} \right]$, with $\boldsymbol{\xi}^{(q_n)} = \left[\begin{array}{ccc} \boldsymbol{\xi}_+^{(q_n)} & \boldsymbol{\xi}_-^{(q_n)} & \boldsymbol{\xi}_0^{(q_n)} \end{array} \right]$ partitioned according to $\mathcal{S} = \mathcal{S}_+ \cup \mathcal{S}_- \cup \mathcal{S}_0$ for $n = 1, \ldots, N$, be the stationary distribution vector of this chain such that $[\boldsymbol{\xi}^{(q_0)}]_i = \lim_{w\to\infty} \mathbb{P}(Y_w^{(1)} = q_0, \varphi_w^{(1)} = i)$ for $i \in \mathcal{S}_-$ and $[\boldsymbol{\xi}^{(q_n)}]_i = \lim_{w\to\infty} \mathbb{P}(Y_w^{(1)} = q_n, \varphi_w^{(1)} = i)$ for $i \in \mathcal{S}$, $n > 0$.

Vector $\boldsymbol{\xi}$ is the unique solution of

$$\begin{aligned} \boldsymbol{\xi}\mathbf{P} &= \boldsymbol{\xi}, & (51)\\ \boldsymbol{\xi}\mathbf{1} &= 1, & (52)\end{aligned}$$

where $\mathbf{P} = \left[\mathbf{P}^{[q_n,q_m]}\right]_{n,m=0,1,\ldots,N}$ is the corresponding one-step transition probability matrix, such that $\mathbf{P}^{[q_n,q_m]} = [P_{ij}^{[q_n,q_m]}]_{i,j\in\mathcal{S}}$, $P_{ij}^{[q_n,q_m]} = \mathbb{P}(Y_{w+1}^{(1)} = q_m, \varphi_{w+1}^{(1)} = j \mid Y_w^{(1)} = q_n, \varphi_w^{(1)} = i)$.

We partition $\boldsymbol{\xi}$ according to $\mathcal{S} = \mathcal{S}_+ \cup \mathcal{S}_- \cup \mathcal{S}_0$, so that

$$\boldsymbol{\xi}^{(q_n)} = \begin{bmatrix} \boldsymbol{\xi}_+^{(q_n)} & \boldsymbol{\xi}_-^{(q_n)} & \boldsymbol{\xi}_0^{(q_n)} \end{bmatrix}, \tag{53}$$

and write explicit expressions for the blocks in $\mathbf{P}$, resulting in

$$\begin{aligned} \mathbf{P}^{[q_0,q_0]} &= \bar{\mathbf{P}}_{-+}^{(q_0)}\mathbf{G}_{+-}(0, q_1 - q_0),\\ \mathbf{P}^{[q_0,q_1]} &= \begin{bmatrix} \bar{\mathbf{P}}_{-+}^{(q_0)}\mathbf{H}_{++}(0, q_1 - q_0) + \mathbf{P}_{-+}^{(q_1)} & \mathbf{P}_{--}^{(q_1)} & \mathbf{P}_{-0}^{(q_1)} \end{bmatrix};\end{aligned}$$

and

$$\mathbf{P}^{[q_1,q_1]} = \begin{bmatrix} \mathbf{0}_{++} & \mathbf{G}_{+-}(0, q_2 - q_1) & \mathbf{0}_{+0}\\ \mathbf{H}_{-+}(q_1 - q_0, q_1 - q_0) & \mathbf{0}_{--} & \mathbf{0}_{-0}\\ (-\mathbf{T}_{00})^{-1}\mathbf{T}_{0-}\mathbf{H}_{-+}(q_1 - q_0, q_1 - q_0) & (-\mathbf{T}_{00})^{-1}\mathbf{T}_{0+}\mathbf{G}_{+-}(0, q_2 - q_1) & \mathbf{0}_{00} \end{bmatrix},$$

$$\mathbf{P}^{[q_1,q_2]} = \begin{bmatrix} \mathbf{H}_{++}(0, q_2 - q_1) & \mathbf{0}_{+-} & \mathbf{0}_{+0}\\ \mathbf{0}_{-+} & \mathbf{0}_{--} & \mathbf{0}_{-0}\\ (-\mathbf{T}_{00})^{-1}\mathbf{T}_{0+}\mathbf{H}_{++}(0, q_2 - q_1) & \mathbf{0}_{0-} & \mathbf{0}_{00} \end{bmatrix},$$

$$\mathbf{P}^{[q_1,q_0]} = \begin{bmatrix} \mathbf{0}_{+-}\\ \mathbf{G}_{--}(q_1 - q_0, q_1 - q_0)\\ (-\mathbf{T}_{00})^{-1}\mathbf{T}_{0-}\mathbf{G}_{--}(q_1 - q_0, q_1 - q_0) \end{bmatrix};$$

and for $q_1 < q_n < q_N$,

$$\mathbf{P}^{[q_0,q_n]} = \begin{bmatrix} \mathbf{P}_{-+}^{(q_n)} & \mathbf{P}_{--}^{(q_n)} & \mathbf{P}_{-0}^{(q_n)} \end{bmatrix},$$

$$\mathbf{P}^{[q_n,q_n]} = \begin{bmatrix} \mathbf{0}_{++} & \mathbf{G}_{+-}(0, q_{n+1} - q_n) & \mathbf{0}_{+0}\\ \mathbf{H}_{-+}(q_n - q_{n-1}, q_n - q_{n-1}) & \mathbf{0}_{--} & \mathbf{0}_{-0}\\ (-\mathbf{T}_{00})^{-1}\mathbf{T}_{0-}\mathbf{H}_{-+}(q_n - q_{n-1}, q_n - q_{n-1}) & (-\mathbf{T}_{00})^{-1}\mathbf{T}_{0+}\mathbf{G}_{+-}(0, q_{n+1} - q_n) & \mathbf{0}_{00} \end{bmatrix},$$

$$\mathbf{P}^{[q_n,q_{n+1}]} = \begin{bmatrix} \mathbf{H}_{++}(0, q_{n+1} - q_n) & \mathbf{0}_{+-} & \mathbf{0}_{+0}\\ \mathbf{0}_{-+} & \mathbf{0}_{--} & \mathbf{0}_{-0}\\ (-\mathbf{T}_{00})^{-1}\mathbf{T}_{0+}\mathbf{H}_{++}(0, q_{n+1} - q_n) & \mathbf{0}_{0-} & \mathbf{0}_{00} \end{bmatrix},$$

$$\mathbf{P}^{[q_n,q_{n-1}]} = \begin{bmatrix} \mathbf{0}_{++} & \mathbf{0}_{+-} & \mathbf{0}_{+0} \\ \mathbf{0}_{-+} & \mathbf{G}_{--}(q_n - q_{n-1}, q_n - q_{n-1}) & \mathbf{0}_{-0} \\ \mathbf{0}_{0+} & (-\mathbf{T}_{00})^{-1}\mathbf{T}_{0-}\mathbf{G}_{--}(q_n - q_{n-1}, q_n - q_{n-1}) & \mathbf{0}_{00} \end{bmatrix};$$

and for $q_N > q_1$,

$$\mathbf{P}^{[q_N,q_N]} = \begin{bmatrix} \mathbf{0}_{++} & \mathbf{\Psi} & \mathbf{0}_{+0} \\ \mathbf{H}_{-+}(q_N - q_{N-1}, q_N - q_{N-1}) & \mathbf{0}_{--} & \mathbf{0}_{-0} \\ (-\mathbf{T}_{00})^{-1}\mathbf{T}_{0-}\mathbf{H}_{-+}(q_N - q_{N-1}, q_N - q_{N-1}) & (-\mathbf{T}_{00})^{-1}\mathbf{T}_{0+}\mathbf{\Psi} & \mathbf{0}_{00} \end{bmatrix},$$

$$\mathbf{P}^{[q_N,q_{N-1}]} = \begin{bmatrix} \mathbf{0}_{++} & \mathbf{0}_{+-} & \mathbf{0}_{+0} \\ \mathbf{0}_{-+} & \mathbf{G}_{--}(q_N - q_{N-1}, q_N - q_{N-1}) & \mathbf{0}_{-0} \\ \mathbf{0}_{0+} & (-\mathbf{T}_{00})^{-1}\mathbf{T}_{0-}\mathbf{G}_{--}(q_N - q_{N-1}, q_N - q_{N-1}) & \mathbf{0}_{00} \end{bmatrix};$$

and $\mathbf{P}^{[q_n,q_m]} = \mathbf{0}$ otherwise.

Denote $\boldsymbol{\pi}_+(q_n^+) = \lim_{x\downarrow q_n} \boldsymbol{\pi}_+(x)$, $\boldsymbol{\pi}_-(q_n^-) = \lim_{x\uparrow q_n} \boldsymbol{\pi}_-(x)$.

Theorem 2. *We have,*

$$\begin{bmatrix} \mathbf{p}_-^{(0)} & \mathbf{p}_0^{(0)} \end{bmatrix} = \alpha\boldsymbol{\xi}_-^{(q_0)}\mathbf{P}_{-\ominus}^{(q_0)}\left(-\mathbf{T}_{\ominus\ominus}\right)^{-1}, \tag{54}$$

$$\boldsymbol{\pi}_+(q_0^+) = \alpha\boldsymbol{\xi}_-^{(q_0)}\bar{\mathbf{P}}_{-+}^{(q_0)}(\mathbf{C}_+)^{-1}; \tag{55}$$

further, for $n > 0$,

$$\mathbf{p}_0^{(n)} = \alpha\boldsymbol{\xi}_0^{(q_n)}\left(-\mathbf{T}_{00}\right)^{-1}, \tag{56}$$

$$\boldsymbol{\pi}_+(q_n^+) = \alpha\boldsymbol{\xi}_+^{(q_n)}(\mathbf{C}_+)^{-1} + \alpha\boldsymbol{\xi}_0^{(q_n)}\left(-\mathbf{T}_{00}\right)^{-1}\mathbf{T}_{0+}(\mathbf{C}_+)^{-1}, \tag{57}$$

$$\boldsymbol{\pi}_-(q_n^-) = \alpha\boldsymbol{\xi}_-^{(q_n)}(|\mathbf{C}_-|)^{-1} + \alpha\boldsymbol{\xi}_0^{(q_n)}\left(-\mathbf{T}_{00}\right)^{-1}\mathbf{T}_{0-}(|\mathbf{C}_-|)^{-1}; \tag{58}$$

for $x \in (q_n, q_{n+1})$, $n = 0, 1, \ldots, N-1$,

$$\begin{aligned} \boldsymbol{\pi}_+(x) =& \boldsymbol{\pi}_+(q_n^+)\mathbf{C}_+\mathbf{N}_{++}^{[b_n]}(0, x - q_n)(\mathbf{C}_+)^{-1} \\ &+ \boldsymbol{\pi}_-(q_{n+1}^-)|\mathbf{C}_-|\mathbf{N}_{-+}^{[b_n]}(q_{n+1} - q_n, x - q_n)(\mathbf{C}_+)^{-1}, \end{aligned} \tag{59}$$

$$\begin{aligned} \boldsymbol{\pi}_-(x) =& \boldsymbol{\pi}_+(q_n^+)\mathbf{C}_+\mathbf{N}_{+-}^{[b_n]}(0, x - q_n)(|\mathbf{C}_-|)^{-1} \\ &+ \boldsymbol{\pi}_-(q_{n+1}^-)|\mathbf{C}_-|\mathbf{N}_{--}^{[b_n]}(q_{n+1} - q_n, x - q_n)(|\mathbf{C}_-|)^{-1}; \end{aligned} \tag{60}$$

for $x > q_N$,

$$\boldsymbol{\pi}_+(x) = \boldsymbol{\pi}_+(q_N^+)\mathbf{C}_+e^{\mathbf{K}(x-q_N)}(\mathbf{C}_+)^{-1}, \tag{61}$$

$$\boldsymbol{\pi}_-(x) = \boldsymbol{\pi}_+(x)\mathbf{C}_+\mathbf{\Psi}(|\mathbf{C}_-|)^{-1}; \tag{62}$$

and for all $x \in (q_n, q_{n+1})$, $n = 0, 1, \ldots, N-1$ *and* $x > q_N$,

$$\boldsymbol{\pi}_0(x) \quad = \quad \left[\begin{array}{cc} \boldsymbol{\pi}_+(x) & \boldsymbol{\pi}_-(x) \end{array}\right] \mathbf{T}_{\pm 0}(-\mathbf{T}_{00})^{-1}; \tag{63}$$

where α *is a normalising constant given by*

$$\begin{aligned}
\alpha = \Bigg\{ & \boldsymbol{\xi}_-^{(q_0)} \mathbf{P}_{-\ominus}^{(q_0)} \left(-\mathbf{T}_{\ominus\ominus}\right)^{-1} \mathbf{1}_{\ominus} + \sum_{n=1}^{N} \boldsymbol{\xi}_0^{(q_n)} \left(-\mathbf{T}_{00}\right)^{-1} \mathbf{1}_0 \\
& + \Bigg(\boldsymbol{\xi}_-^{(q_0)} \bar{\mathbf{P}}_{-+}^{(q_0)} \left[\begin{array}{cc} \int_{x=q_0}^{q_1} \mathbf{N}_{++}^{[b_0]}(0, x-q_0) dx (\mathbf{C}_+)^{-1} & \int_{x=q_0}^{q_1} \mathbf{N}_{+-}^{[b_0]}(0, x-q_0) dx (|\mathbf{C}_-|)^{-1} \end{array}\right] \\
& + \left(\boldsymbol{\xi}_-^{(q_1)} + \boldsymbol{\xi}_0^{(q_1)} \left(-\mathbf{T}_{00}\right)^{-1} \mathbf{T}_{0-} \right) \\
& \quad \times \left[\begin{array}{cc} \int_{x=q_0}^{q_1} \mathbf{N}_{-+}^{[b_0]}(q_1-q_0, x-q_0) dx (\mathbf{C}_+)^{-1} & \int_{x=q_0}^{q_1} \mathbf{N}_{--}^{[b_0]}(q_1-q_0, x-q_0) dx (|\mathbf{C}_-|)^{-1} \end{array}\right] \\
& + \sum_{n=1}^{N-1} \left(\boldsymbol{\xi}_+^{(q_n)} + \boldsymbol{\xi}_0^{(q_n)} \left(-\mathbf{T}_{00}\right)^{-1} \mathbf{T}_{0+} \right) \\
& \quad \times \left[\begin{array}{cc} \int_{x=q_n}^{q_{n+1}} \mathbf{N}_{++}^{[b_n]}(0, x-q_n) dx (\mathbf{C}_+)^{-1} & \int_{x=q_n}^{q_{n+1}} \mathbf{N}_{+-}^{[b_n]}(0, x-q_n) dx (|\mathbf{C}_-|)^{-1} \end{array}\right] \\
& + \sum_{n=1}^{N-1} \left(\boldsymbol{\xi}_-^{(q_{n+1})} + \boldsymbol{\xi}_0^{(q_{n+1})} \left(-\mathbf{T}_{00}\right)^{-1} \mathbf{T}_{0-} \right) \\
& \quad \times \left[\begin{array}{cc} \int_{x=q_n}^{q_{n+1}} \mathbf{N}_{-+}^{[b_n]}(q_{n+1}-q_n, x-q_n) dx (\mathbf{C}_+)^{-1} & \int_{x=q_n}^{q_{n+1}} \mathbf{N}_{--}^{[b_n]}(q_{n+1}-q_n, x-q_n) dx (|\mathbf{C}_-|)^{-1} \end{array}\right] \\
& + \left(\boldsymbol{\xi}_+^{(q_N)} + \boldsymbol{\xi}_0^{(q_N)} \left(-\mathbf{T}_{00}\right)^{-1} \mathbf{T}_{0+} \right) \\
& \quad \times (-\mathbf{K})^{-1} (\mathbf{C}_+)^{-1} \left[\begin{array}{cc} \mathbf{I}_+ & \mathbf{C}_+ \boldsymbol{\Psi} (|\mathbf{C}_-|)^{-1} \end{array}\right] \Bigg) \\
& \times \left(\mathbf{1}_{\pm} + \mathbf{T}_{\pm 0} (-\mathbf{T}_{00})^{-1} \mathbf{1}_0 \right) \Bigg\}^{-1},
\end{aligned}$$

and for all n, *denoting* $q = q_{n+1} - q_n$,

$$\begin{aligned}
& \int_{x=q_n}^{q_{n+1}} \mathbf{N}_{++}^{[b_n]}(0, x-q_n) dx \\
= \quad & (\mathbf{I} - e^{\mathbf{K}q} \boldsymbol{\Psi} e^{\mathbf{J}q} \boldsymbol{\Xi})^{-1} \left((\mathbf{K})^{-1} (e^{\mathbf{K}q} - \mathbf{I}) - e^{\mathbf{K}q} \boldsymbol{\Psi} \left(\int_{z=0}^{q} e^{\mathbf{J}z} dz \right) \boldsymbol{\Xi} \right), \\
& \int_{x=q_n}^{q_{n+1}} \mathbf{N}_{+-}^{[b_n]}(0, x-q_n) dx \\
= \quad & (\mathbf{I} - e^{\mathbf{K}q} \boldsymbol{\Psi} e^{\mathbf{J}q} \boldsymbol{\Xi})^{-1} \left((\mathbf{K})^{-1} (e^{\mathbf{K}q} - \mathbf{I}) \boldsymbol{\Psi} - e^{\mathbf{K}q} \boldsymbol{\Psi} \left(\int_{z=0}^{q} e^{\mathbf{J}z} dz \right) \right),
\end{aligned}$$

$$\begin{aligned}
&\int_{x=q_n}^{q_{n+1}} \mathbf{N}_{-+}^{[b_n]}(q_{n+1}-q_n, x-q_n)dx \\
= \quad &(\mathbf{I} - e^{\mathbf{J}q}\boldsymbol{\Xi}e^{\mathbf{K}q}\boldsymbol{\Psi})^{-1}\left(-e^{\mathbf{J}q}\boldsymbol{\Xi}(\mathbf{K})^{-1}(e^{\mathbf{K}q}-\mathbf{I}) + \left(\int_{z=0}^{q} e^{\mathbf{J}z}dz\right)\boldsymbol{\Xi}\right), \\
&\int_{x=q_n}^{q_{n+1}} \mathbf{N}_{--}^{[b_n]}(q_{n+1}-q_n, x-q_n)dx \\
= \quad &(\mathbf{I} - e^{\mathbf{J}q}\boldsymbol{\Xi}e^{\mathbf{K}q}\boldsymbol{\Psi})^{-1}\left(-e^{\mathbf{J}q}\boldsymbol{\Xi}(\mathbf{K})^{-1}(e^{\mathbf{K}q}-\mathbf{I})\boldsymbol{\Psi} + \left(\int_{z=0}^{q} e^{\mathbf{J}z}dz\right)\right).
\end{aligned}$$

Proof. The first three equations follow by conditioning similar to Theorem 1. Next, (57) follows since there are two alternative ways for the process to leave level $q_n > 0$ in some phase in $\mathcal{S}_+$,

- by a visit to level q_n from below in some phase in $\mathcal{S}_+$ (the first term in (57)), or
- by a visit to level q_n in some phase in $\mathcal{S}_0$ (immediately after a jump to level zero), followed by a transition to some phase in $\mathcal{S}_+$ after spending some time on level q_n (the second term in (57)).

We obtain (58) by a symmetrical argument.

Further, (59) follows by conditioning on the last visit to q_n or q_{n+1}, since for the process to be observed at level $x \in (q_n, q_{n+1})$,

- it must either leave level q_n in some phase in $\mathcal{S}_+$ and then visit x any number of times, ending in some phase in $\mathcal{S}_+$ (the first term in (59)), or
- leave level q_{n+1} in some phase in $\mathcal{S}_-$ and then visit x any number of times, ending in some phase in $\mathcal{S}_+$ (the second term in (59)).

Then, (60) follows by a similar conditioning.

Next, α is a normalising constant which guarantees that

$$\sum_{n=0,1,\ldots,N} \mathbf{p}^{(n)}\mathbf{1} + \sum_{n=0,1,\ldots,N} \int_{x=q_n}^{q_{n+1}} \boldsymbol{\pi}(x)dx\mathbf{1} \quad = \quad 1. \tag{64}$$

Denoting $q = q_{n+1} - q_n$, by (34) we have,

$$\begin{aligned}
&\int_{x=q_n}^{q_{n+1}} \mathbf{N}_{++}^{[b_n]}(0, x-q_n)dx \\
&= \int_{x=0}^{q} \mathbf{N}_{++}^{[q]}(0, x)dx \\
&= \int_{x=0}^{q} (\mathbf{I} - e^{\mathbf{K}q}\boldsymbol{\Psi}e^{\mathbf{J}q}\boldsymbol{\Xi})^{-1}(e^{\mathbf{K}x} - e^{\mathbf{K}q}\boldsymbol{\Psi}e^{\mathbf{J}(q-x)}\boldsymbol{\Xi})dx \\
&= (\mathbf{I} - e^{\mathbf{K}q}\boldsymbol{\Psi}e^{\mathbf{J}q}\boldsymbol{\Xi})^{-1}\left((\mathbf{K})^{-1}(e^{\mathbf{K}q}-\mathbf{I}) - e^{\mathbf{K}q}\boldsymbol{\Psi}\left(\int_{z=0}^{q} e^{\mathbf{J}z}dz\right)\boldsymbol{\Xi}\right)
\end{aligned}$$

and

$$\int_{x=q_n}^{q_{n+1}} \mathbf{N}_{-+}^{[b_n]}(q_{n+1}-q_n, x-q_n)dx$$
$$= \int_{x=0}^{q} \mathbf{N}_{-+}^{[q]}(q,x)dx$$
$$= \int_{x=0}^{q} (\mathbf{I} - e^{\mathbf{J}q}\Xi e^{\mathbf{K}q}\mathbf{\Psi})^{-1}(-e^{\mathbf{J}q}\Xi e^{\mathbf{K}x} + e^{\mathbf{J}(q-x)}\Xi)dx$$
$$= (\mathbf{I} - e^{\mathbf{J}q}\Xi e^{\mathbf{K}q}\mathbf{\Psi})^{-1}\left(-e^{\mathbf{J}q}\Xi(\mathbf{K})^{-1}(e^{\mathbf{K}q}-\mathbf{I}) + \left(\int_{z=0}^{q} e^{\mathbf{J}z}dz\right)\Xi\right),$$

since

$$\int_{x=0}^{q} e^{\mathbf{K}x}dx = (\mathbf{K})^{-1}\left[e^{\mathbf{K}x}\right]_{x=0}^{q} = (\mathbf{K})^{-1}\left(e^{\mathbf{K}q}-\mathbf{I}\right),$$

and with $z = q - x$, $dz = -dx$, we get

$$\int_{x=0}^{q} e^{\mathbf{J}(q-x)}dx = \int_{z=q}^{0} -e^{\mathbf{J}z}dz = \int_{z=0}^{q} e^{\mathbf{J}z}dz.$$

Expressions for $\int_{x=q_n}^{q_{n+1}} \mathbf{N}_{-+}^{[b_n]}(q_{n+1}-q_n, x-q_n)dx$ and $\int_{x=q_n}^{q_{n+1}} \mathbf{N}_{-+}^{[b_n]}(q_{n+1}-q_n, x-q_n)dx$ follow in an analogous manner.

3.4. Embedded Markov chain #2: conditioning on leaving levels q_n

It is also of interest to consider a discrete-time Markov chain (labelled #2) observed at the times the process *leaves* one of the levels q_n, $n = 0, 1, \ldots, N$, due to the fluid level increasing or decreasing in some phase in $\mathcal{S}_+ \cup \mathcal{S}_-$. The stationary probability of such chain records proportion of times the SFM leaves levels q_n, and does so in some phase in $\mathcal{S}_+$ when leaving level q_0, or in some phase in $\mathcal{S}_+ \cup \mathcal{S}_-$ when leaving level q_n for $n = 1, \ldots, N$.

So, consider a discrete-time Markov chain $\{(Y_w^{(2)}, \varphi_t^{(2)}) : w = 0, 1, \ldots\}$ with state space $(\{0\} \times \mathcal{S}_+) \cup (\{1, \ldots, N\} \times (\mathcal{S}_+ \cup \mathcal{S}_-))$ observed at the times when the SFM $\{(X(t), J(t)) : t \geq 0\}$ *leaves* one of the levels q_n, $n = 0, 1, \ldots, N$.

Let $\boldsymbol{\xi}_\pm = \left[\boldsymbol{\xi}_+(q_0), \boldsymbol{\xi}_\pm(q_1), \ldots, \boldsymbol{\xi}_\pm(q_N)\right]$, with $\boldsymbol{\xi}_\pm(q_n) = \left[\begin{array}{cc} \boldsymbol{\xi}_+(q_n) & \boldsymbol{\xi}_-(q_n) \end{array}\right]$ partitioned according to $\mathcal{S}_+ \cup \mathcal{S}_-$ for $n > 0$, be the stationary distribution vector of this chain such that $\left[\boldsymbol{\xi}_+(q_0)\right]_i = \lim_{w\to\infty} \mathbb{P}(Y_w^{(2)} = q_0, \varphi_w^{(2)} = i)$ for $i \in \mathcal{S}_+$, and $\left[\boldsymbol{\xi}_\pm(q_n)\right]_i = \lim_{w\to\infty} \mathbb{P}(Y_w^{(2)} = q_n, \varphi_w^{(2)} = i)$ for $i \in \mathcal{S}_+ \cup \mathcal{S}_-$, $n > 0$.

Vector $\boldsymbol{\xi}_\pm$ is the unique solution of

$$\boldsymbol{\xi}_\pm \mathbf{P}_{\pm\pm} = \boldsymbol{\xi}_\pm, \tag{65}$$

$$\boldsymbol{\xi}_{\pm}\mathbf{1} \;=\; 1, \tag{66}$$

where $\mathbf{P}_{\pm\pm} = \left[\mathbf{P}_{\pm\pm}^{[q_n,q_m]}\right]_{n,m=0,1,\ldots,N}$ is the corresponding one-step transition probability matrix, such that $\mathbf{P}_{\pm\pm}^{[q_n,q_m]} = [P_{ij}^{[q_n,q_m]}]_{i,j\in\mathcal{S}}$, $P_{ij}^{[q_n,q_m]} = \mathbb{P}(Y_{w+1}^{(2)} = q_m, \varphi_{w+1}^{(2)} = j \mid Y_w^{(2)} = q_n, \varphi_w^{(2)} = i)$.

We write explicit expressions for the blocks in $\mathbf{P}_{\pm\pm}$, resulting in

$$\mathbf{P}_{\pm\pm}^{[q_0,q_0]} = \left[\begin{array}{cc} \mathbf{G}_{+-}(0,q_1-q_0)\bar{\mathbf{P}}_{-+}^{(q_9)} & \mathbf{0}_{+-} \end{array}\right],$$

$$\mathbf{P}_{\pm\pm}^{[q_0,q_1]} = \left[\begin{array}{cc} \mathbf{H}_{++}(0,q_1-q_0)+\mathbf{G}_{+-}(0,q_1-q_0)\bar{\mathbf{P}}_{-+}^{(q_1)} & \mathbf{G}_{+-}(0,q_1-q_0)\bar{\mathbf{P}}_{--}^{(q_1)} \end{array}\right],$$

$$\mathbf{P}_{\pm\pm}^{[q_1,q_0]} = \left[\begin{array}{c} \mathbf{0}_{++} \\ \mathbf{G}_{--}(q_1-q_0,q_1-q_0)\bar{\mathbf{P}}_{-+}^{(q_0)}\mathbf{0}_{--} \end{array}\right],$$

$$\mathbf{P}_{\pm\pm}^{[q_1,q_1]} = \left[\begin{array}{cc} \mathbf{0}_{++} & 1\{q_1<q_N\}\mathbf{G}_{+-}(0,q_2-q_1)+1\{q_1=q_N\}\boldsymbol{\Psi} \\ \mathbf{H}_{-+}(q_1-q_0,q_1-q_0)+\mathbf{G}_{--}(q_1-q_0,q_1-q_0)\bar{\mathbf{P}}_{-+}^{(q_1)} & \mathbf{G}_{--}(q_1-q_0,q_1-q_0)\bar{\mathbf{P}}_{--}^{(q_1)} \end{array}\right].$$

$$\mathbf{P}_{\pm\pm}^{[q_1,q_2]} = \left[\begin{array}{cc} \mathbf{H}_{++}(0,q_2-q_1) & \mathbf{0}_{+-} \\ \mathbf{G}_{--}(q_1-q_0,q_1-q_0)\bar{\mathbf{P}}_{-+}^{(q_2)} & \mathbf{G}_{--}(q_1-q_0,q_1-q_0)\bar{\mathbf{P}}_{--}^{(q_2)} \end{array}\right];$$

and for $q_1 < q_n < q_N$,

$$\begin{aligned}
\mathbf{P}_{\pm\pm}^{[q_0,q_n]} &= \left[\begin{array}{cc} \mathbf{G}_{+-}(0,q_1-q_0)\bar{\mathbf{P}}_{-+}^{(q_n)} & \mathbf{G}_{+-}(0,q_1-q_0)\bar{\mathbf{P}}_{--}^{(q_n)} \end{array}\right], \\
\mathbf{P}_{\pm\pm}^{[q_n,q_n]} &= \left[\begin{array}{cc} \mathbf{0}_{++} & \mathbf{G}_{+-}(0,q_n-q_{n-1}) \\ \mathbf{H}_{-+}(q_n-q_{n-1},q_n-q_{n-1}) & \mathbf{0}_{--} \end{array}\right], \\
\mathbf{P}_{\pm\pm}^{[q_n,q_{n+1}]} &= \left[\begin{array}{cc} \mathbf{H}_{++}(0,q_{n+1}-q_n) & \mathbf{0}_{+-} \\ \mathbf{0}_{-+} & \mathbf{0}_{--} \end{array}\right], \\
\mathbf{P}_{\pm\pm}^{[q_n,q_{n-1}]} &= \left[\begin{array}{cc} \mathbf{0}_{++} & \mathbf{0}_{+-} \\ \mathbf{0}_{-+} & \mathbf{G}_{--}(q_n-q_{n-1},q_n-q_{n-1}) \end{array}\right];
\end{aligned}$$

and for $q_N > q_1$,

$$\begin{aligned}
\mathbf{P}_{\pm\pm}^{[q_N,q_N]} &= \left[\begin{array}{cc} \mathbf{0}_{++} & \boldsymbol{\Psi} \\ \mathbf{H}_{-+}(q_N-q_{N-1},q_N-q_{N-1}) & \mathbf{0}_{--} \end{array}\right], \\
\mathbf{P}_{\pm\pm}^{[q_N,q_{N-1}]} &= \left[\begin{array}{cc} \mathbf{0}_{++} & \mathbf{0}_{+-} \\ \mathbf{0}_{-+} & \mathbf{G}_{--}(q_N-q_{N-1}) \end{array}\right];
\end{aligned}$$

and $\mathbf{P}_{\pm\pm}^{[q_n,q_m]} = \mathbf{0}$ otherwise.

We now consider the case with no probability masses at levels q_n, $n = 0, 1, \ldots, N$.

Theorem 3. *Suppose* $\mathbf{P}_{-\ominus}^{(q_n)}\mathbf{1} = \mathbf{0}$ *for all* $n \geq 0$*. Then,*

$$\boldsymbol{\pi}_+(q_0^+) \;=\; \alpha\boldsymbol{\xi}_+(q_0)(\mathbf{C}_+)^{-1}, \tag{67}$$

and for $n = 1, \ldots, N$*,*

$$\begin{bmatrix} \boldsymbol{\pi}_+(q_n^+) & \boldsymbol{\pi}_-(q_n^-) \end{bmatrix} \;=\; \alpha\boldsymbol{\xi}_\pm(q_n)(|\mathbf{C}_\pm|)^{-1}. \tag{68}$$

For $x \in (q_n, q_{n+1})$ *for some* $n = 0, \ldots, N-1$*,*

$$\begin{aligned} \boldsymbol{\pi}_+(x) \;=\;& \boldsymbol{\pi}_+(q_n^+)\mathbf{C}_+\mathbf{N}_{++}(0, x - q_n)(\mathbf{C}_+)^{-1} \\ &+ \boldsymbol{\pi}_-(q_{n+1}^-)|\mathbf{C}_-|\mathbf{N}_{-+}(q_{n+1} - q_n, x - q_n)(\mathbf{C}_+)^{-1}. \end{aligned} \tag{69}$$

For $x > q_N$*,*

$$\boldsymbol{\pi}_+(x) \;=\; \boldsymbol{\pi}_+(q_N^+)\mathbf{C}_+e^{\mathbf{K}(x-q_N)}(\mathbf{C}_+)^{-1}. \tag{70}$$

For all $x > 0$*,*

$$\boldsymbol{\pi}_-(x) \;=\; \boldsymbol{\pi}_+(x)\mathbf{C}_+\boldsymbol{\Psi}(|\mathbf{C}_-|)^{-1}, \tag{71}$$

$$\boldsymbol{\pi}_0(x) \;=\; \begin{bmatrix} \boldsymbol{\pi}_+(x) & \boldsymbol{\pi}_-(x) \end{bmatrix}\mathbf{T}_{\pm 0}(-\mathbf{T}_{00})^{-1}. \tag{72}$$

The normalising constant is given by

$$\begin{aligned} \alpha =& \Bigg\{ \Bigg(\sum_{n=0}^{N-1} \int_{x=q_n}^{q_{n+1}} \Big(\boldsymbol{\xi}_+(q_n)\mathbf{C}_+\mathbf{N}_{++}^{[b_n]}(0, x - q_n)(\mathbf{C}_+)^{-1} \\ &+ \boldsymbol{\xi}_-(q_{n+1})|\mathbf{C}_-|\mathbf{N}_{-+}^{[b_n]}(q_{n+1} - q_n, x - q_n)(\mathbf{C}_+)^{-1}\Big) + \boldsymbol{\pi}_+(q_N)\mathbf{C}_+(-\mathbf{K})^{-1} \Bigg) \\ &\times \Big(\mathbf{1}_+ + \mathbf{C}_+\boldsymbol{\Psi}(|\mathbf{C}_-|)^{-1}\mathbf{1}_- + \begin{bmatrix} \mathbf{I}_+ & \mathbf{C}_+\boldsymbol{\Psi}(|\mathbf{C}_-|)^{-1} \end{bmatrix} \mathbf{T}_{\pm 0}(-\mathbf{T}_{00})^{-1}\mathbf{1}_0 \Big) dx \Bigg\}^{-1}. \end{aligned} \tag{73}$$

Proof. Equation (68) follows by the uniqueness of $\boldsymbol{\xi}_\pm$ and by noticing that $[\boldsymbol{\pi}_\pm(q_n)|\mathbf{C}_\pm|]_{n=0,1,\ldots,N}$, with $\boldsymbol{\pi}_\pm(q_n) = \begin{bmatrix} \boldsymbol{\pi}_+(q_n) & \boldsymbol{\pi}_-(q_n) \end{bmatrix}$, is a solution of (65). Equation (69) follows by conditioning of the process leaving level q_n in some phase in $\mathcal{S}_+$ or q_{n+1} in some phase in $\mathcal{S}_-$. Equations (70)-(72) follow similarly to Lemma 1.

Next,

$$
\begin{aligned}
1 =& \int_{x=0}^{\infty} \boldsymbol{\pi}(x)dx\mathbf{1} \\
=& \Bigg(\sum_{n=0}^{N-1} \int_{x=q_n}^{q_{n+1}} \Big(\boldsymbol{\pi}_+(q_n)\mathbf{C}_+\mathbf{N}_{++}^{[b_n]}(0, x-q_n)(\mathbf{C}_+)^{-1} \\
&+ \boldsymbol{\pi}_-(q_{n+1})|\mathbf{C}_-|\mathbf{N}_{-+}^{[b_n]}(q_{n+1}-q_n, x-q_n)(\mathbf{C}_+)^{-1}\Big)\, dx \\
&+ \int_{x=q_N}^{\infty} \boldsymbol{\pi}_+(q_N)\mathbf{C}_+e^{\mathbf{K}(x-q_N)}(\mathbf{C}_+)^{-1}dx \Bigg) \\
&\times \Big(\mathbf{1}_+ + \mathbf{C}_+\boldsymbol{\Psi}(|\mathbf{C}_-|)^{-1}\mathbf{1}_- + \left[\begin{array}{cc} \mathbf{I}_+ & \mathbf{C}_+\boldsymbol{\Psi}(|\mathbf{C}_-|)^{-1} \end{array} \right] \mathbf{T}_{\pm 0}(-\mathbf{T}_{00})^{-1}\mathbf{1}_0 \Big) \\
=& \alpha \Bigg(\sum_{n=0}^{N-1} \int_{x=q_n}^{q_{n+1}} \Big(\boldsymbol{\xi}_+(q_n)\mathbf{C}_+\mathbf{N}_{++}^{[b_n]}(0, x-q_n)(\mathbf{C}_+)^{-1} \\
&+ \boldsymbol{\xi}_-(q_{n+1})|\mathbf{C}_-|\mathbf{N}_{-+}^{[b_n]}(q_{n+1}-q_n, x-q_n)(\mathbf{C}_+)^{-1}\Big)\, dx + \boldsymbol{\xi}_+(q_N)(-\mathbf{K})^{-1}(\mathbf{C}_+)^{-1} \Bigg) \\
&\Big(\mathbf{1}_+ + \mathbf{C}_+\boldsymbol{\Psi}(|\mathbf{C}_-|)^{-1}\mathbf{1}_- + \left[\begin{array}{cc} \mathbf{I}_+ & \mathbf{C}_+\boldsymbol{\Psi}(|\mathbf{C}_-|)^{-1} \end{array} \right] \mathbf{T}_{\pm 0}(-\mathbf{T}_{00})^{-1}\mathbf{1}_0 \Big)
\end{aligned}
$$

and the expression for α follows by the above.

4. Transient Analysis of SFMs with Upward Jumps and Phase Transitions

Suppose that the process starts from some level $X(0) = z > 0$ in some phase $J(0) = i \in \mathcal{S}$ according to the initial distribution vector $\boldsymbol{\alpha}(z) = [\alpha_i(z)]_{i\in\mathcal{S}}$, with $\alpha_i(z) = \mathbb{P}(J(0) = i \mid X(0) = z)$.

Consider the LSTs ${}_{\boldsymbol{\alpha}(z)}\mathcal{L}_{q_n}^{(R)}(s)$, $n = 0, 1, \ldots, N$, such that

$$
\left[{}_{\boldsymbol{\alpha}(z)}\mathcal{L}_{q_n}^{(R)}(s)\right]_j = \int_{t=0}^{\infty} {}_{\boldsymbol{\alpha}(z)}\mathbb{E}(e^{-sR(t)}1\{X(t) = q_n, J(t) = j\})dt
$$

is the LST of the total reward/cost accumulated when observing the process at level q_n in phase j, given the initial distribution; and the LSTs ${}_{\boldsymbol{\alpha}(z)}\mathcal{L}_{q_n}(s)$ such that

$$
\begin{aligned}
{}_{\boldsymbol{\alpha}(z)}\mathcal{L}_{q_n}(s) &= \int_{t=0}^{\infty} e^{-st}{}_{\boldsymbol{\alpha}(z)}\mathbf{p}^{(n)}(t)dt \\
\left[{}_{\boldsymbol{\alpha}(z)}\mathbf{p}^{(n)}(t)\right]_i &= {}_{\boldsymbol{\alpha}(z)}\mathbb{P}(X(t) = q_n, J(t) = i).
\end{aligned}
$$

Also, consider the LSTs ${}_{\boldsymbol{\alpha}(z)}\mathcal{L}_{\mathbf{f}}^{(R)}(x; s)$ for $x > 0$, such that

$$
\left[{}_{\boldsymbol{\alpha}(z)}\mathcal{L}_{\mathbf{f}}^{(R)}(x; s)\right]_j = \int_{t=0}^{\infty} {}_{\boldsymbol{\alpha}(z)}\mathbb{E}(e^{-sR(t)}1\{X(t) = x, J(t) = j\})dt
$$

is the LST of the total reward/cost accumulated when observing the process at level x in some phase in $\mathcal{S}$, given the initial distribution; and the LSTs ${}_{\boldsymbol{\alpha}(z)}\mathcal{L}_{\mathbf{f}}(x;s)$ for $x>0$, defined as

$$ {}_{\boldsymbol{\alpha}(z)}\mathcal{L}_{\mathbf{f}}(x;s)=\int_{t=0}^{\infty}e^{-st}{}_{\boldsymbol{\alpha}(z)}\mathbf{f}(t,x)dt=\int_{t=0}^{\infty}e^{-st}\left[\begin{array}{ccc} {}_{\boldsymbol{\alpha}(z)}\mathbf{f}(t,x)_{+} & {}_{\boldsymbol{\alpha}(z)}\mathbf{f}(t,x)_{-} & {}_{\boldsymbol{\alpha}(z)}\mathbf{f}(t,x)_{0}\end{array}\right]dt, $$

$$ \left[{}_{\boldsymbol{\alpha}(z)}\mathbf{f}(t,x)\right]_i=\frac{\partial}{\partial x}\,{}_{\boldsymbol{\alpha}(z)}\mathbb{P}(X(t)\le x, J(t)=i). $$

Below we derive the results for ${}_{\boldsymbol{\alpha}(z)}\mathcal{L}_{q_n}^{(R)}(s)$ and ${}_{\boldsymbol{\alpha}(z)}\mathcal{L}_{\mathbf{f}}^{(R)}(x;s)$. We note that if $r_i=1$ for all $i\in\mathcal{S}$, then ${}_{\boldsymbol{\alpha}(z)}\mathcal{L}_{q_n}(s)={}_{\boldsymbol{\alpha}(z)}\mathcal{L}_{q_n}^{(R)}(s)$ and ${}_{\boldsymbol{\alpha}(z)}\mathcal{L}_{\mathbf{f}}(x;s)={}_{\boldsymbol{\alpha}(z)}\mathcal{L}_{\mathbf{f}}^{(R)}(x;s)$, and so the expressions for ${}_{\boldsymbol{\alpha}(z)}\mathcal{L}_{q_n}(s)$ and ${}_{\boldsymbol{\alpha}(z)}\mathcal{L}_{\mathbf{f}}(x;s)$ follow by letting $\mathbf{R}=\mathbf{I}$.

Denote

$$ \begin{aligned} \mathbf{P}_{--}^{(R)}(s)&=\bar{\mathbf{P}}_{-+}^{(q_0)}(s)\boldsymbol{\Psi}^{(R)}(s)\\ &\quad+\sum_{n=1}^{N}\left(\bar{\mathbf{P}}_{--}^{(q_n)}(s)+\bar{\mathbf{P}}_{-+}^{(q_n)}(s)\boldsymbol{\Psi}^{(R)}(s)\right)e^{\left(\mathbf{W}_{--}(s)+\mathbf{W}_{-+}(s)\boldsymbol{\Psi}^{(R)}(s)\right)q_n}, \end{aligned} \tag{74} $$

recording the LSTs of the reward/cost accumulated during paths contributing to $\mathbf{P}_{--}=\mathbf{P}_{--}^{(R)}(0)$ defined in (37), and the LSTs of the time corresponding to these paths in the case if $r_i=1$ for all $i\in\mathcal{S}$.

Further, let

$$ \begin{aligned} \bar{\boldsymbol{\alpha}}_{+}(s)&=\boldsymbol{\alpha}_{+}(z)+\boldsymbol{\alpha}_{0}(z)(s\mathbf{R}_0-\mathbf{T}_{00})^{-1}\mathbf{T}_{0+},\\ \bar{\boldsymbol{\alpha}}_{-}(s)&=\boldsymbol{\alpha}_{-}(z)+\boldsymbol{\alpha}_{0}(z)(s\mathbf{R}_0-\mathbf{T}_{00})^{-1}\mathbf{T}_{0-}. \end{aligned} $$

Theorem 4. *Assume that* $\chi\left(s\mathbf{R}_0-\mathbf{T}_{00}\right)<0$. *For* $x\in(q_n,q_{n+1})$, $n=0,1\ldots,N-1$, *and* $x>q_N$, *we have*

$$ \begin{aligned} {}_{\boldsymbol{\alpha}(z)}\mathcal{L}_{\mathbf{f}}^{(R)}(x;s)_{+}=\Bigg\{&\left(\bar{\boldsymbol{\alpha}}_{+}(s)\boldsymbol{\Psi}^{(R)}(s)+\bar{\boldsymbol{\alpha}}_{-}(s)\right)e^{\mathbf{D}^{(R)}(s)z}\left(\mathbf{I}-\mathbf{P}_{--}^{(R)}(s)\right)^{-1}\\ &\times\Bigg(\bar{\mathbf{P}}_{-+}^{(q_0)}(s)e^{\mathbf{K}^{(R)}(s)x}\\ &\quad+\sum_{q_n:0<q_n<x}\left(\bar{\mathbf{P}}_{--}^{(q_n)}(s)\mathbf{H}_{-+}^{(R)(q_n,q_n)}(s)+\bar{\mathbf{P}}_{-+}^{(q_n)}(s)\right)\\ &\quad\mathbf{H}_{++}^{(R)(q_n,x)}(s)\left(\mathbf{I}-\boldsymbol{\Psi}^{(R)}(s)\mathbf{H}_{-+}^{(R)(x,x)}(s)\right)^{-1}\\ &\quad+\sum_{q_n:q_n>x}\left(\bar{\mathbf{P}}_{--}^{(q_n)}(s)+\bar{\mathbf{P}}_{-+}^{(q_n)}(s)\boldsymbol{\Psi}^{(R)}(s)\right) \end{aligned} $$

$$
\begin{aligned}
& \qquad e^{\mathbf{D}^{(R)}(s)(q_n-x)}\mathbf{H}_{-+}^{(R)(x,x)}(s)\left(\mathbf{I}-\mathbf{\Psi}^{(R)}(s)\mathbf{H}_{-+}^{(R)(x,x)}(s)\right)^{-1}\Bigg) \\
& + 1\{x<z\}\left(\bar{\boldsymbol{\alpha}}_{+}(s)\mathbf{\Psi}^{(R)}(s)+\bar{\boldsymbol{\alpha}}_{-}(s)\right) \\
& \qquad e^{\mathbf{D}^{(R)}(s)(z-x)}\mathbf{H}_{-+}^{(R)(x,x)}(s)\left(\mathbf{I}-\mathbf{\Psi}^{(R)}(s)\mathbf{H}_{-+}^{(R)(x,x)}(s)\right)^{-1} \\
& + 1\{x>z\}\left(\bar{\boldsymbol{\alpha}}_{+}(s)+\bar{\boldsymbol{\alpha}}_{-}(s)\mathbf{H}_{-+}^{(R)(z,z)}(s)\right) \\
& \qquad \mathbf{H}_{++}^{(R)(z,x)}(s)\left(\mathbf{I}-\mathbf{\Psi}^{(R)}(s)\mathbf{H}_{-+}^{(R)(x,x)}(s)\right)^{-1}\Bigg\}(\mathbf{C}_{+})^{-1}, \qquad (75)
\end{aligned}
$$

and

$$
\begin{aligned}
{}_{\boldsymbol{\alpha}(z)}\mathcal{L}_{\mathbf{f}}^{(R)}(x;s)_{-} = \Bigg\{ & \left(\bar{\boldsymbol{\alpha}}_{+}(s)\mathbf{\Psi}^{(R)}(s)+\bar{\boldsymbol{\alpha}}_{-}(s)\right)e^{\mathbf{D}^{(R)}(s)z}\left(\mathbf{I}-\mathbf{P}_{--}^{(R)}(s)\right)^{-1} \\
& \times\Bigg(\bar{\mathbf{P}}_{-+}^{(q_0)}(s)e^{\mathbf{K}^{(R)}(s)x}\mathbf{\Psi}^{(R)}(s) \\
& \quad + \sum_{q_n:0<q_n<x}\left(\bar{\mathbf{P}}_{--}^{(q_n)}(s)\mathbf{H}_{-+}^{(R)(q_n,q_n)}(s)+\bar{\mathbf{P}}_{-+}^{(q_n)}(s)\right) \\
& \qquad \mathbf{H}_{++}^{(R)(q_n,x)}(s)\left(\mathbf{I}-\mathbf{\Psi}^{(R)}(s)\mathbf{H}_{-+}^{(R)(x,x)}(s)\right)^{-1}\mathbf{\Psi}^{(R)}(s) \\
& \quad + \sum_{q_n:q_n>x}\left(\bar{\mathbf{P}}_{--}^{(q_n)}(s)+\bar{\mathbf{P}}_{-+}^{(q_n)}(s)\mathbf{\Psi}^{(R)}(s)\right) \\
& \qquad e^{\mathbf{D}^{(R)}(s)(q_n-x)}\left(\mathbf{I}-\mathbf{H}_{-+}^{(R)(x,x)}(s)\mathbf{\Psi}^{(R)}(s)\right)^{-1}\Bigg) \\
& + 1\{x<z\}\left(\bar{\boldsymbol{\alpha}}_{+}(s)\mathbf{\Psi}^{(R)}(s)+\bar{\boldsymbol{\alpha}}_{-}(s)\right) \\
& \quad e^{\mathbf{D}^{(R)}(s)(z-x)}\left(\mathbf{I}-\mathbf{H}_{-+}^{(R)(x,x)}(s)\mathbf{\Psi}^{(R)}(s)\right)^{-1} \\
& + 1\{x>z\}\left(\bar{\boldsymbol{\alpha}}_{+}(s)+\bar{\boldsymbol{\alpha}}_{-}(s)\mathbf{H}_{-+}^{(R)(z,z)}(s)\right) \\
& \quad \mathbf{H}_{++}^{(R)(z,x)}(s)\left(\mathbf{I}-\mathbf{\Psi}^{(R)}(s)\mathbf{H}_{-+}^{(R)(x,x)}(s)\right)^{-1}\mathbf{\Psi}^{(R)}(s)\Bigg\}(|\mathbf{C}_{-}|)^{-1}, \qquad (76)
\end{aligned}
$$

and

$$
{}_{\boldsymbol{\alpha}(z)}\mathcal{L}_{\mathbf{f}}^{(R)}(x;s)_{0} \quad = \quad \left[\; {}_{\boldsymbol{\alpha}(z)}\mathcal{L}_{\mathbf{f}}^{(R)}(x;s)_{+} \quad {}_{\boldsymbol{\alpha}(z)}\mathcal{L}_{\mathbf{f}}^{(R)}(x;s)_{-} \;\right]\mathbf{T}_{\pm 0}(s\mathbf{R}_0-\mathbf{T}_{00})^{-1}. \qquad (77)
$$

Further, if $\mathbf{P}_{-\ominus}^{(q_0)}\mathbf{1} \neq \mathbf{0}$*, then* $\mathbf{p}^{(0)}(t) \neq \mathbf{0}$ *exists and*

$$_{\boldsymbol{\alpha}(z)}\mathcal{L}_{q_0}^{(R)}(s) = \left(\bar{\boldsymbol{\alpha}}_+(s)\boldsymbol{\Psi}^{(R)}(s) + \bar{\boldsymbol{\alpha}}_-(s)\right) e^{\mathbf{D}^{(R)}(s)z}\left(\mathbf{I} - \mathbf{P}_{--}^{(R)}(s)\right)^{-1}\mathbf{P}_{-\ominus}^{(q_0)}(s\mathbf{R}_{\ominus} - \mathbf{T}_{\ominus\ominus})^{-1}, \tag{78}$$

and, if $\mathbf{P}_{-\ominus}^{(q_n)}\mathbf{1} \neq \mathbf{0}$*, then* $\mathbf{p}^{(n)}(t) \neq \mathbf{0}$ *exists and*

$$_{\boldsymbol{\alpha}(z)}\mathcal{L}_{q_n}^{(R)}(s) = \left(\bar{\boldsymbol{\alpha}}_+(s)\boldsymbol{\Psi}^{(R)}(s) + \bar{\boldsymbol{\alpha}}_-(s)\right) e^{\mathbf{D}^{(R)}(s)z}\left(\mathbf{I} - \mathbf{P}_{--}^{(R)}(s)\right)^{-1}\mathbf{P}_{-0}^{(q_n)}(s\mathbf{R}_0 - \mathbf{T}_{00})^{-1}. \tag{79}$$

Proof. The result follows by the usual level-crossing arguments, similar to Theorem 1. The first four lines in (75) correspond to the conditioning that in order to observe the process at level x at time t in some phase in $\mathcal{S}_+$, the process may visit level zero before visiting level x. To do so, the fluid must first drain to level zero and then visit level zero again any number of times (first line in (75)). Next, one of the three alternatives may occur.

- The process may reflect from level zero and then visit level x (second line in (75)).
- The process may jump to some level $q_n < x$ and then visit level x (third line in (75)).
- The process may jump to some level $q_n > x$ and then visit level x (fourth line in (75)).

The last two lines in (75) follow by conditioning that the process may visit level x without visiting level zero. Expression (76) follows by analogous arguments.

Expression (78) follows by conditioning that in order to observe the process at level zero in some phase in $\mathcal{S}_- \cup \mathcal{S}_0$ with some probability at time t, the process must drain from level zero, then transition to some phase in $\mathcal{S}_- \cup \mathcal{S}_0$, and then remain at level zero for some time.

Expression (79) follows by conditioning that in order to observe the process at level $q_n > 0$ in some phase in $\mathcal{S}_0$ with some probability at time t, the process must drain from level zero, then jump to level q_n in some phase in $\mathcal{S}_0$, and then remain at level q_n for some time.

We also state the result for the model studied by Kulkarni and Yan in [19], as a special case of the model considered here.

Corollary 5. *Suppose* $\mathbf{P}^{(q_0)}\mathbf{1} = \mathbf{0}$ *and* $N = 1$*. Denote* $q = q_N$*. Assume that* $\chi\left(s\mathbf{R}_0 - \mathbf{T}_{00}\right) < 0$.

Then, for $x > 0$*,*

$$_{\boldsymbol{\alpha}(z)}\mathcal{L}_{\mathbf{f}}^{(R)}(x;s)_+ = \Bigg\{ \left(\bar{\boldsymbol{\alpha}}_+(s)\boldsymbol{\Psi}^{(R)}(s) + \bar{\boldsymbol{\alpha}}_-(s)\right) e^{\mathbf{D}^{(R)}(s)z}\left(\mathbf{I} - \mathbf{P}_{--}^{(R)}(s)\right)^{-1}$$
$$\times \Bigg(1\{q_1 < x\}\left(\bar{\mathbf{P}}_{--}^{(q_1)}(s)\mathbf{H}_{-+}^{(R)(q_1,q_1)}(s) + \bar{\mathbf{P}}_{-+}^{(q_1)}(s)\right)$$

$$\mathbf{H}_{++}^{(R)(q_1,x)}(s)\left(\mathbf{I}-\mathbf{\Psi}^{(R)}(s)\mathbf{H}_{-+}^{(R)(x,x)}(s)\right)^{-1}$$

$$+1\{q_1>x\}\left(\bar{\mathbf{P}}_{--}^{(q_1)}(s)+\bar{\mathbf{P}}_{-+}^{(q_1)}(s)\mathbf{\Psi}^{(R)}(s)\right)$$

$$\left. e^{\mathbf{D}^{(R)}(s)(q_1-x)}\mathbf{H}_{-+}^{(R)(x,x)}(s)\left(\mathbf{I}-\mathbf{\Psi}^{(R)}(s)\mathbf{H}_{-+}^{(R)(x,x)}(s)\right)^{-1}\right)$$

$$+1\{x<z\}\left(\bar{\boldsymbol{\alpha}}_+(s)\mathbf{\Psi}^{(R)}(s)+\bar{\boldsymbol{\alpha}}_-(s)\right)$$

$$e^{\mathbf{D}^{(R)}(s)(z-x)}\mathbf{H}_{-+}^{(R)(x,x)}(s)\left(\mathbf{I}-\mathbf{\Psi}^{(R)}(s)\mathbf{H}_{-+}^{(R)(x,x)}(s)\right)^{-1}$$

$$+1\{x>z\}\left(\bar{\boldsymbol{\alpha}}_+(s)+\bar{\boldsymbol{\alpha}}_-(s)\mathbf{H}_{-+}^{(R)(z,z)}(s)\right)$$

$$\left. \mathbf{H}_{++}^{(R)(z,x)}(s)\left(\mathbf{I}-\mathbf{\Psi}^{(R)}(s)\mathbf{H}_{-+}^{(R)(x,x)}(s)\right)^{-1}\right\}(\mathbf{C}_+)^{-1},$$

and

$$\begin{aligned}
{}_{\boldsymbol{\alpha}(z)}\mathcal{L}_{\mathbf{f}}^{(R)}(x;s)_- = \Bigg\{ & \left(\bar{\boldsymbol{\alpha}}_+(s)\mathbf{\Psi}^{(R)}(s)+\bar{\boldsymbol{\alpha}}_-(s)\right)e^{\mathbf{D}^{(R)}(s)z}\left(\mathbf{I}-\mathbf{P}_{--}^{(R)}(s)\right)^{-1} \\
& \times\Bigg(1\{q_1<x\}\left(\bar{\mathbf{P}}_{--}^{(q_1)}(s)\mathbf{H}_{-+}^{(R)(q_1,q_1)}(s)+\bar{\mathbf{P}}_{-+}^{(q_1)}(s)\right) \\
& \quad \mathbf{H}_{++}^{(R)(q_1,x)}(s)\left(\mathbf{I}-\mathbf{\Psi}^{(R)}(s)\mathbf{H}_{-+}^{(R)(x,x)}(s)\right)^{-1}\mathbf{\Psi}^{(R)}(s) \\
& \quad +1\{q_1>x\}\left(\bar{\mathbf{P}}_{--}^{(q_1)}(s)+\bar{\mathbf{P}}_{-+}^{(q_1)}(s)\mathbf{\Psi}^{(R)}(s)\right) \\
& \quad e^{\mathbf{D}^{(R)}(s)(q_1-x)}\left(\mathbf{I}-\mathbf{H}_{-+}^{(R)(x,x)}(s)\mathbf{\Psi}^{(R)}(s)\right)^{-1}\Bigg) \\
& +1\{x<z\}\left(\bar{\boldsymbol{\alpha}}_+(s)\mathbf{\Psi}^{(R)}(s)+\bar{\boldsymbol{\alpha}}_-(s)\right) \\
& \quad e^{\mathbf{D}^{(R)}(s)(z-x)}\left(\mathbf{I}-\mathbf{H}_{-+}^{(R)(x,x)}(s)\mathbf{\Psi}^{(R)}(s)\right)^{-1} \\
& +1\{x>z\}\left(\bar{\boldsymbol{\alpha}}_+(s)+\bar{\boldsymbol{\alpha}}_-(s)\mathbf{H}_{-+}^{(R)(z,z)}(s)\right) \\
& \quad \mathbf{H}_{++}^{(R)(z,x)}(s)\left(\mathbf{I}-\mathbf{\Psi}^{(R)}(s)\mathbf{H}_{-+}^{(R)(x,x)}(s)\right)^{-1}\mathbf{\Psi}^{(R)}(s)\Bigg\}(|\mathbf{C}_-|)^{-1},
\end{aligned}$$

and

$$ {}_{\boldsymbol{\alpha}(z)}\mathcal{L}_{\mathbf{f}}^{(R)}(x;s)_0 = \left[\ {}_{\boldsymbol{\alpha}(z)}\mathcal{L}_{\mathbf{f}}^{(R)}(x;s)_+ \quad {}_{\boldsymbol{\alpha}(z)}\mathcal{L}_{\mathbf{f}}^{(R)}(x;s)_-\ \right]\mathbf{T}_{\pm 0}(s\mathbf{R}_0-\mathbf{T}_{00})^{-1},$$

where

$$\mathbf{P}_{--}^{(R)}(s) \;=\; \left(\bar{\mathbf{P}}_{--}^{(q)}(s) + \bar{\mathbf{P}}_{-+}^{(q)}(s)\boldsymbol{\Psi}^{(R)}(s)\right) e^{\mathbf{D}^{(R)}(s)q}.$$

5. Examples

Example 1. Assume $\mathcal{S} = \{1,2,3,4,5,6\}$, $|\mathcal{S}_+| = 3$, $|\mathcal{S}_-| = 2$, $|\mathcal{S}_0| = 1$, $[c_i]$ =$[1,2,3,-4,$ $-3,0]$, such that

$$\mathbf{T} = \begin{bmatrix} -6 & 1 & 1 & 4 & 0 & 0 \\ 1 & -12 & 1 & 4 & 5 & 1 \\ 1 & 1 & -12 & 4 & 5 & 1 \\ 0.8 & 0.8 & 0.8 & -6.4 & 3.2 & 0.8 \\ 0.8 & 0.8 & 1,6 & 3.2 & -7.2 & 0.8 \\ 1 & 1 & 1 & 3.2 & 4 & -10.2 \end{bmatrix}$$

and $N = 2$, $q_1 = 2$, $q_2 = 3$, with $\mathbf{P}^{(q_0)}\mathbf{1} = (1/2)\mathbf{1}$, $\mathbf{P}^{(q_1)}\mathbf{1} = (1/3)\mathbf{1}$, $\mathbf{P}^{(q_2)}\mathbf{1} = (1/6)\mathbf{1}$, such that $[\mathbf{P}^{(q_0)}]_{ij} = (1/2)(1/6)$, $[\mathbf{P}^{(q_1)}]_{ij} = (1/3)(1/6)$, $[\mathbf{P}^{(q_2)}]_{ij} = (1/6)(1/6)$ for all i,j.

We apply results in Section 3 to evaluate the stationary distribution of the process. This gives

$$\begin{aligned} \begin{bmatrix} \mathbf{p}_-^{(0)} & \mathbf{p}_0^{(0)} \end{bmatrix} &= \begin{bmatrix} 0.0394 & 0.0375 & 0.0145 \end{bmatrix}, \\ \mathbf{p}_0^{(1)} &= 0.0056, \\ \mathbf{p}_0^{(2)} &= 0.0028, \end{aligned}$$

with the probability of observing level zero equal to $\mathbb{P}(X=0) = \sum_i p_i^{(0)} = \mathbf{p}^{(0)}\mathbf{1} = 0.0914$, and the probability of observing levels above zero equal to $\mathbb{P}(X > 0) = 1 - \sum_i p_i^{(0)} = 1 - \mathbf{p}^{(0)}\mathbf{1} = 0.9086$. The plot of the densities $\pi_i(x)$ is displayed in Figure 3.

The stationary distributions $\boldsymbol{\xi}_-^{(0)}, \boldsymbol{\xi} = \left[\boldsymbol{\xi}_-^{(q_0)}, \boldsymbol{\xi}^{(q_1)}, \ldots, \boldsymbol{\xi}^{(q_N)}\right]$, and $\boldsymbol{\xi}_\pm = \big[\boldsymbol{\xi}_+(q_0), \boldsymbol{\xi}_\pm(q_1),$ $\ldots, \boldsymbol{\xi}_\pm(q_N)\big]$ of Markov chains #0, #1, and #2, recording proportions of visits to phases in $\mathcal{S}_-$ when visiting level q_0; proportion of visits to levels q_n; and proportion of times the process leaves levels q_n; respectively, are

$$\boldsymbol{\xi}_-^{(0)} \;=\; \begin{bmatrix} 0.6131 & 0.3869 \end{bmatrix},$$

and

$$\boldsymbol{\xi}_-^{(q_0)} = \begin{bmatrix} 0.2603 & 0.1643 \end{bmatrix}, \quad \boldsymbol{\xi}_-^{(q_0)}\mathbf{1} = 0.4246,$$

$$\begin{aligned}
\left[\begin{array}{ccc} \boldsymbol{\xi}_+^{(q_1)} & \boldsymbol{\xi}_-^{(q_1)} & \boldsymbol{\xi}_0^{(q_1)} \end{array}\right] &= \left[\begin{array}{cccccc} 0.0383 & 0.0394 & 0.0543 & 0.1510 & 0.1060 & 0.0236 \end{array}\right], \\
\boldsymbol{\xi}^{(q_1)}\mathbf{1} &= 0.4126, \\
\left[\begin{array}{ccc} \boldsymbol{\xi}_+^{(q_2)} & \boldsymbol{\xi}_-^{(q_2)} & \boldsymbol{\xi}_0^{(q_2)} \end{array}\right] &= \left[\begin{array}{cccccc} 0.0179 & 0.0184 & 0.0256 & 0.0513 & 0.0378 & 0.0118 \end{array}\right], \\
\boldsymbol{\xi}^{(q_2)}\mathbf{1} &= 0.1628,
\end{aligned}$$

and

$$\begin{aligned}
\boldsymbol{\xi}_+(q_0) &= \left[\begin{array}{ccc} 0.0893 & 0.0893 & 0.1058 \end{array}\right], \quad \boldsymbol{\xi}_+(q_0)\mathbf{1} = 0.2843, \\
\left[\begin{array}{cc} \boldsymbol{\xi}_+(q_1) & \boldsymbol{\xi}_-(q_1) \end{array}\right] &= \left[\begin{array}{ccccc} 0.0545 & 0.0560 & 0.0762 & 0.2065 & 0.1604 \end{array}\right], \\
\boldsymbol{\xi}_{+-}(q_1)\mathbf{1} &= 0.5535, \\
\left[\begin{array}{cc} \boldsymbol{\xi}_+(q_2) & \boldsymbol{\xi}_-(q_2) \end{array}\right] &= \left[\begin{array}{ccccc} 0.0187 & 0.0190 & 0.0220 & 0.0571 & 0.0454 \end{array}\right], \\
\boldsymbol{\xi}_{+-}(q_2)\mathbf{1} &= 0.1622,
\end{aligned}$$

respectively.

Next, suppose the process starts from level $z = 1$, and

$$\boldsymbol{\alpha}(z) \quad = \quad [\alpha_i(z)]_{i\in\mathcal{S}} = \left[\begin{array}{cccccc} 1/6 & 1/6 & 1/6 & 1/4 & 1/4 & 0 \end{array}\right].$$

We evaluate the distribution of the process at time t using Theorem 4, and two alternative LST inversion algorithms (to verify our computations), one by Horváth et al. [18], and another by Den Iseger [14] as coded in Toutain et al. [28]. The results are presented in Figures 4-5.

We note that the output $\mathbf{p}^{(n)}(t)$ and $\boldsymbol{\pi}(x,t)$ for large t from the transient analysis, agrees with the output $\mathbf{p}^{(n)}$ and $\boldsymbol{\pi}(x)$ from the stationary analysis, as expected.

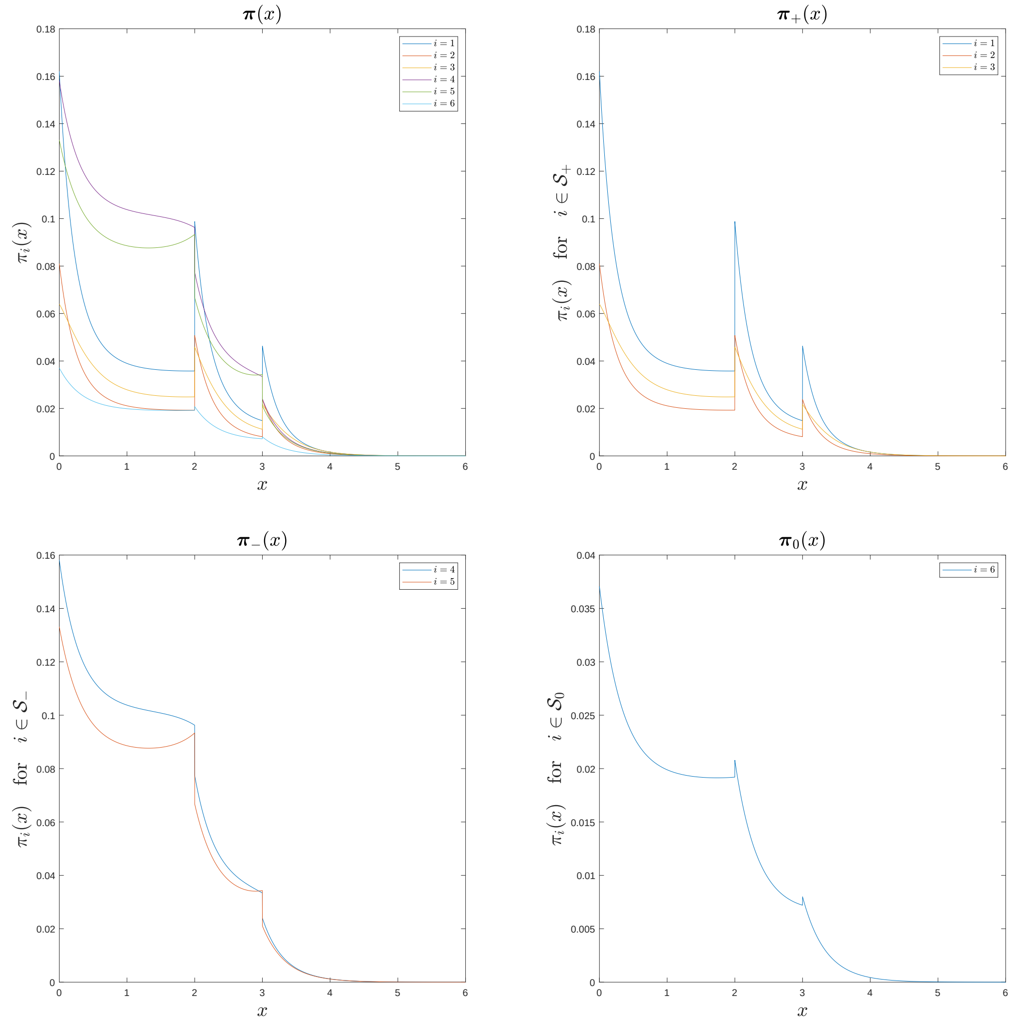

Figure 3. Stationary distribution of the process $\{(X(t), J(t)) : t \geq 0\}$ in Example 1: $\boldsymbol{\pi}(x)$ for $x \in (q_0, q_1)$, $x \in (q_1, q_2)$, and $x > q_2$, where $q_0 = 0$, $q_1 = 2$, $q_2 = 3$. We observe discontinuities in $\boldsymbol{\pi}(x)$ at levels $x = q_1, q_2$ due to nonzero probabilities $\mathbf{p}_0^{(n)}$ for $n = 1, 2$. This occurs since the process jumps to levels q_n in some phases in $\mathcal{S}_0$, and then remains at these levels until it transitions to some phase in $\mathcal{S}_+ \cup \mathcal{S}_-$.

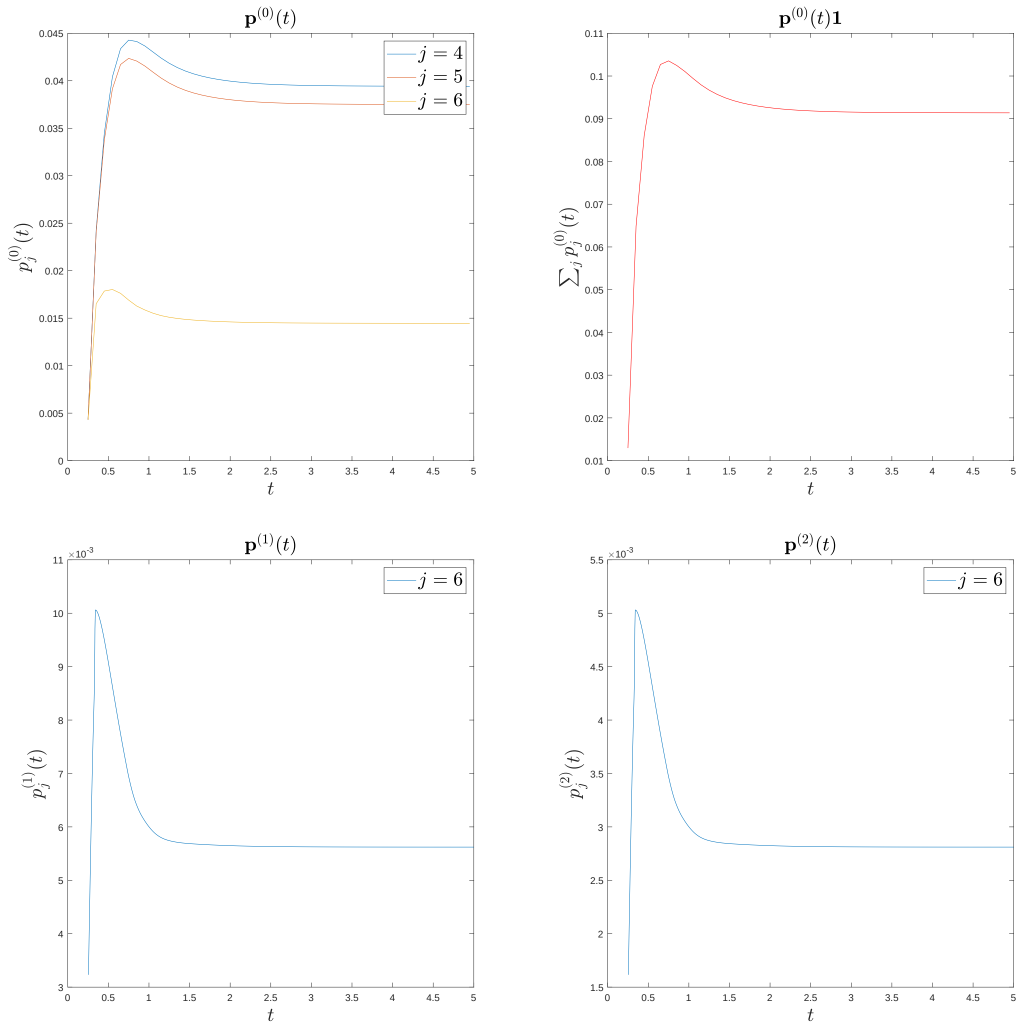

Figure 4. Distribution at time t of the process $\{(X(t), J(t)) : t \geq 0\}$ in Example 1: $\mathbf{p}^{(n)}(t)$, $n = 0, 1, 2$. We observe the convergence of $\mathbf{p}^{(n)}(t)$ to the stationary distribution $\mathbf{p}^{(n)}$ for all n as expected. We have $\mathbf{p}^{(n)}(t) = \mathbf{0}$ for all $t < z/|\min\{c_i\}_{i\in\mathcal{S}}| = 0.25$, which is the minimum time for the process to reach level q_n (minimum time required to drain from level z to level zero and then jump to level q_n).

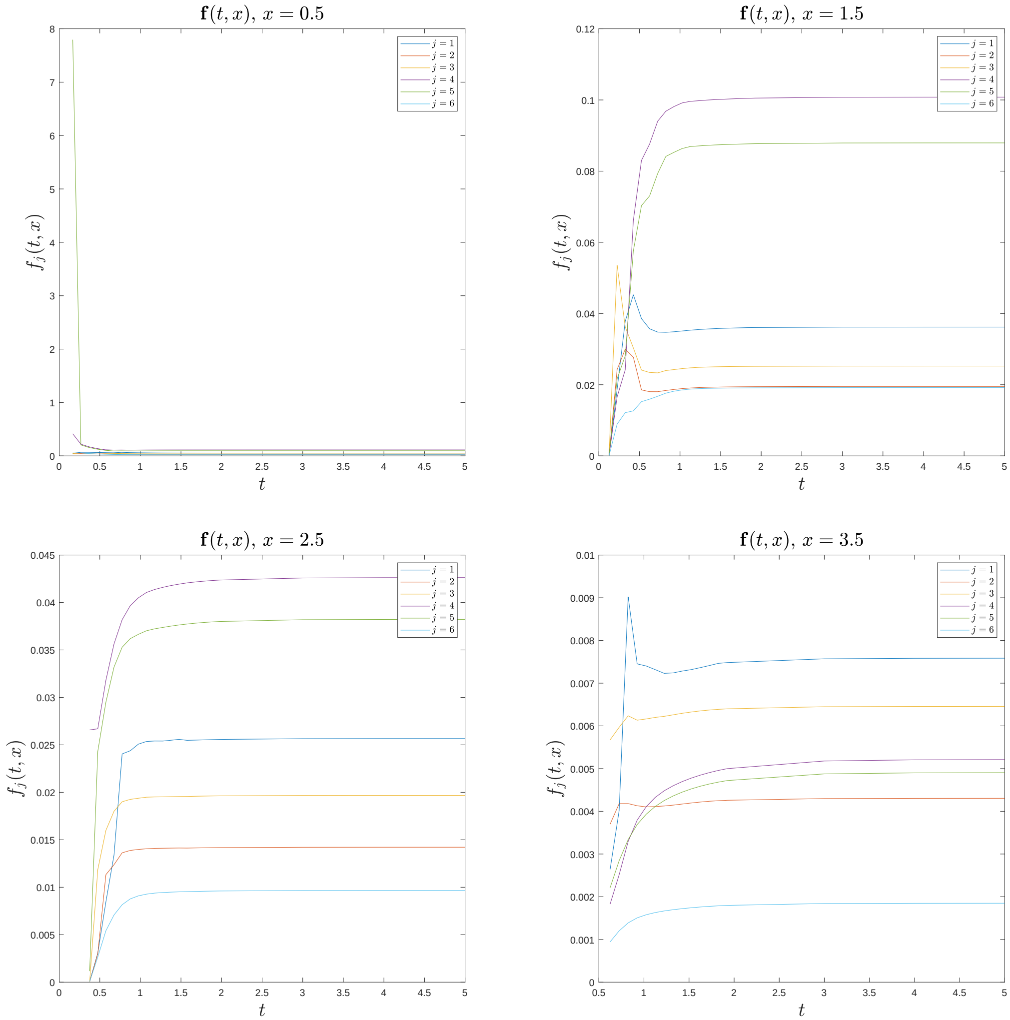

Figure 5. Distribution at time t of the process $\{(X(t), J(t)) : t \geq 0\}$ in Example 1: $\mathbf{f}(x,t)$ for $x = 0.5, 1.5, 2.5, 3.5$. We observe the convergence of $\mathbf{f}(x,t)$ to the stationary distribution $\boldsymbol{\pi}(x)$ as expected.

Example 2. Now, suppose that a reward/cost is accumulated at rates $[r_i] = [1, 3, 2, 0, 0, 0]$ in the SFM in Example 1. The stationary distribution $\boldsymbol{\nu} = [\nu_i]$ of the background phase process $\{J(t) : t \geq 0\}$ given by

$$\begin{aligned}\boldsymbol{\nu} &= \left[\begin{array}{ccc} \boldsymbol{\nu}_+ & \boldsymbol{\nu}_- & \boldsymbol{\nu}_0 \end{array}\right] \\ &= \left[\begin{array}{ccc} \int_0^\infty \boldsymbol{\pi}_+(x)dx & \mathbf{p}_-^{(0)} + \int_0^\infty \boldsymbol{\pi}_-(x)dx & \sum_{n=0}^N \mathbf{p}_0^{(n)} + \int_0^\infty \boldsymbol{\pi}_0(x)dx \end{array}\right],\end{aligned}$$

here is equal to

$$\boldsymbol{\nu} \;=\; \left[\begin{array}{cccccc} 0.1506 & 0.0811 & 0.0981 & 0.3128 & 0.2767 & 0.0807 \end{array}\right], \tag{80}$$

and so the long run expected reward/cost rate is $\sum_i \nu_i r_i = 0.5901$.

Also, note that the stationary distribution $\boldsymbol{\gamma} = [\gamma_i]$ of a continuous-time Markov chain with state space $\mathcal{S}$ and generator $\mathbf{T}$, which is the solution of $\boldsymbol{\gamma}\mathbf{T} = \mathbf{0}$, $\boldsymbol{\gamma}\mathbf{1} = 1$, is equal to

$$\boldsymbol{\gamma} \;=\; \begin{bmatrix} 0.1241 & 0.0668 & 0.0853 & 0.3564 & 0.3010 & 0.0665 \end{bmatrix} \neq \boldsymbol{\nu},$$

where the inequality is due to the fact that the stationary distribution of the phase process is affected by the phase transitions at the moments of jumps in the SFM.

The distribution of the reward/cost accumulated during various sample paths of the SFM can be evaluated by inverting the corresponding LSTs. As example, to compute the distribution of the reward/cost accumulated between two successive visits to level 0, we invert the LST matrix $\mathbf{P}^{(R)}_{--}(s)$. The output is presented in Figure 6.

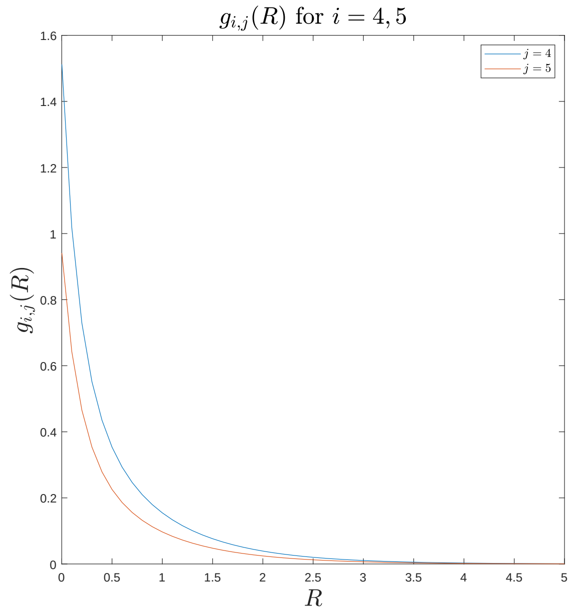

Figure 6. Distribution of the total reward/cost R accumulated between two successive visits to level 0 in Example 1: $\mathbf{g}(R) = [g_{i,j}(R)]$ for $i, j = 4, 5$ records the probability density the total reward/cost R accumulated at the time of the visit to level 0 and doing so in phase j, given start from level 0 in phase i.

6. Conclusion

We have studied a stochastic fluid model (SFM) $\{(X(t), J(t)) : t \geq 0\}$ with upward jumps to levels $q_n > 0$, $n = 1, \ldots, N$, and phase transitions, at the times of visiting level

$q_0 = 0$. We derived results for the stationary and transient analysis of the process through the application of matrix-analytic methods (MAMs), and illustrated the application potential of our methodology through numerical examples.

As far as the authors are aware, the process was previously analysed using various algebraic methods in a special case with $N = 1$, and without special behaviour at the boundary $q_0 = 0$ (which would allow the process to remain at level zero or reflect from it). We have analysed this process using fluid generators [5, 8, 9, 27], together with conditioning on the evolution of the sample paths of the SFM, level-crossing arguments, and useful physical interpretations, illustrating the convenient techniques available in the theory of MAMs [15, 21].

As has been discussed in the literature [7, 19, 23], this class of models has application potential to a wide range of problems that are of practical interest in real-world systems. For example, the matrices that record the probabilities of phase transitions at the times of jump can be used to model the management of systems in which we may wish to control how the systems behave [7]. We will report related extensions and analyses in our forthcoming and future work.

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