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# The Non-parametric Asymmetric Kernel Method in the Study of M/G/1 Queue with Optional Second Service

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**Abstract:** This paper proposes the non-parametric asymmetric kernel method in the study of M/G/1 queue with optional second service. In this model customers arriving following a Poisson process and all demand the first service, while only some of them demand the second service with probability $p$. The service times are assumed to follow a general distribution. We introduce the modified Gamma (MG) kernel method considered as a good alternative to parametric approaches such as mixture models for approximating the service and re-service time densities. Some performance measures related to this system are illustrated through simulation studies.

**Keywords:** MG kernel, M/G/1 queue, non-parametric density estimation, optional service, simulation.

## 1. Introduction

Queuing systems with optional second service are of particular interest and constitute an important application area, see for example, [10, 22, 11, 12, 21], etc. The emphasis on these systems is due to their uses in everyday life. Indeed, in general consider customers depart the system after taking their service. But in many applied queuing systems, some customers need to be re-serviced after taking their main service. For example, in production and manufacturing lines, in hospital, computer and communication systems, etc.

The model we have chosen to analyze is an M/G/1 queuing system with optional second service. In this queueing system, customers arrive at the system one by one according to a Poisson stream with mean arrival rate $\lambda$. There is a single server who provides the first essential service to all arriving customers according to a general distribution. As soon as the first service of a customer is complete, then with probability $p$ he may opt for the second service, in which case his second service will immediately commence or else with probability $1-p$ he may opt to leave the system. The second service times as assumed to be general.

This queueing system has previously been studied by [18, 19] from classical queuing theory aspect, they considered three disciplines for re-servicing in this queuing system and

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obtained the probability generating function (pgf) of the steady-state system size at the moment of departure of the customer in the main queue, the mean busy period and the probability of the idle period. Also, for this system [14] develop a Bayesian framework using a mixture of truncated Normal distributions [14]. Later, [15] extended this approach by employing a mixture of Gamma distributions to approximate general service and re-service times distributions in queueing systems within a Bayesian approach. It is noting that the Bayesian analysis of queueing systems in general is a relatively recent field of research; for example [1, 2, 20] focus on M/M/1 queues, [16] on an M/G/1 queues, [4] on M/G/1 queue using a phase-type approximation, and [3] on G/M/c queues, etc. The studies by [14, 15] can be viewed as an extension of the previously mentioned works, as they incorporate the re-service aspect into their analysis.

Although this queuing system has been studied by [15], where they considered a mixture model for approximating the general distributions. We propose a non-parametric model for modeling the general time distributions of service and re-service. Notice that, in many practical situations, parametric or semi-parametric life models can be performed poorly. As good alternative to these approach, is the non-parametric kernel method, which does not make hypothesis on the form of the general distributions to be estimated, and is easy to implement in practice.

Thus, in this work we estimate the general function of service and re-service time densities using the modified Gamma (MG) kernel method, where the general law of service and re-service G is unknown so its density function can be estimated by using the non-parametric kernel density method. This approach has numerous advantages. First, it imposes no parametric restrictions in the form of the curve. Second, it has good statistical properties such as bias, variance and mean integrated squared error. As a third advantage, the kernel density estimator provides a smoother density estimate than the histogram, see [9], [23].

The application of non-parametric kernel methods in queueing systems remains relatively recent but increasingly promising. To the best of our knowledge, there are two studies on the use of kernel methods in queueing theory. For example, [5] investigated the strong stability of the M/M/1 queue using kernel density estimation, while [8] applied asymmetric kernel methods in the analysis of the strong stability of the PH/M/1 queue. In the same spirit as the previous works, our study contributes to extending non-parametric kernel estimation techniques to more complex queuing models, specifically the M/G/1 system with optional second service. This offers modeling flexibility and better represents the variability in real service processes.

The main contribution of this paper is to introduce non-parametric model for the general density of service and re-service. We employed the MG kernel method to estimate the service and re-service times densities when the underlying distribution $G$ is unknown. Some performance measures related to this queuing system such as the mean system size, the mean busy period and probability of idle period have been estimated. The approach is illustrated with several numerical examples based on various simulation studies.

The remainder of this paper is organized as follows. Section 2 we describe an M/G/1 queuing system with optional second service. In section 3 we define our approach based on non-parametric MG kernel method to the study of the $M/G/1$ queuing system with optional second service. Section 4 gives some numerical results. Section 5 gives conclusions and possible extensions.

## 2. The M/G/1 Queuing System with Optional Second Service

### *2.1. Model's description*

We consider an $M/G/1$ queue with second optional service, in which some items failed and needs to be reserved with probability "$p$", independent of its service time or other factors (queuing system with two queues and one server). Moreover, we consider this system under a specific policy. We assume failed items are stockpiled in a failed queue (FQ) and re-serviced only after all customers in the main queue (MQ) are serviced. After completion of re-service of all items in FQ, the server returns to MQ if there are any customers waiting in MQ; otherwise, the server is idle. We assume that customers arrive to the system according to a Poisson process with rate $\lambda$. For service times, we suppose that service and re-service times are independent and have general distributions, denoted by $F_1(.)$ and $F_2(.)$ with means $\mu_1$, $\mu_2$ respectively.

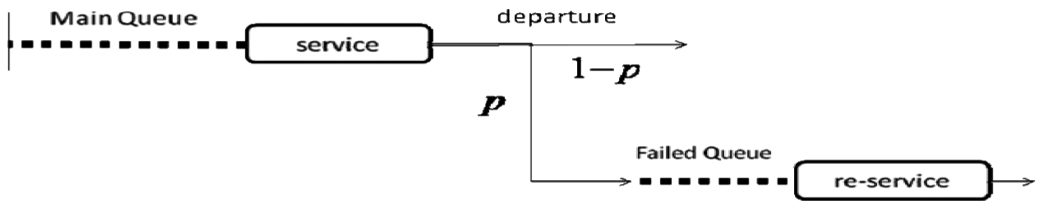


Figure 1. M/G/1 queuing system in which some items failed with probability $p$, and required re-service.

### *2.2. Performance measures of the model*

For our queuing system, we assume that the traffic intensity $\rho$, is less than one and the queuing system is in equilibrium.
The traffic intensity $\rho$ is given by: $\rho = \rho_1 + p\rho_2$, in which $\rho_1 = \lambda\mu_1$ is traffic intensity in MQ and $\rho_2 = \lambda\mu_2$ is traffic intensity in FQ, [15].
Under this steady state condition, other performance measures will be obtained.

- The Mean Busy Period and Probability of Idle Period of the System

$$\mathbb{E}[busy\ period] = \frac{\mu_1 + p^2\mu_2}{1 - \lambda\mu_1},$$

and

$$P[idle\ period] = \frac{1 - \lambda\mu_1}{1 + p^2\lambda\mu_2}.$$

- The Mean System Size

Suppose that $X_n$ is the number of customers remaining in MQ at the completion of the nth customer's service time also $Y_n$ is the number of customers remaining in FQ at the completion of the nth customer's service time in the steady state. The expression of the mean system size is given by:

$$\mathbb{E}(X_n) = \rho_1 + \frac{\lambda^2\delta_1 + \rho_1^2}{2(1-\rho_1)} + \frac{p[\lambda^2\delta_2 + \rho_2^2 + p\rho_2^2(\frac{\lambda^2\delta_1+\rho_1(2-\rho_1)}{1-\rho_1})]}{2(p\rho_2 + (1-\rho_1)\Psi(1-p+pB_2^*(\lambda)))},$$

and

$$\mathbb{E}(Y_n) = \frac{p(2(1-\rho_1)^2 + \lambda^2\delta_1 + \rho_1(1-\rho_1))}{2(p\rho_2 + (1-\rho_1)\Psi(1-p+pB_2^*(\lambda)))}.$$

Where, $B_1^*(.)$ and $B_2^*(.)$ are the Laplace Stieltjes Transform (LST) of the service and re-service times density, and $\delta_1$, $\delta_2$ are the variance of service and re-service times, respectively.

### *2.3. Mixture models and Bayesian analysis: a brief review*

In this section, we briefly review the Mohammadi and Salhi-Rad 's works. [14] exploit the Bayesian inference and prediction for an $M/G/1$ queuing model with optional second re-service. They introduced a mixture model for the service and re-service distributions based on a mixture of truncated Normal distributions on interval $(0,\infty)$ and developed the Bayesian approach for approximating the general distribution of service and re-service based on former articles. This queuing system has also been studied by [15] where they use a mixture of Gamma distributions, providing a Bayesian approach for approximating the general distributions in queuing systems based on former work. The authors introduce a Bayesian algorithm based on the birth-death MCMC approach in order to fit this model to data, see [14] and [15]. Some important measures, such as the system size mean, the mean busy period and probability of idle period have been estimated. This methodology has been illustrated with simulation study. The `bmixture` package based on the Bayesian estimates for the finite mixture of distributions is also implemented in the statistical R software, see [13].

## 3. Non-parametric Asymmetric Kernel Method in the Study of $M/G/1$ Queuing System with Optional Second Service

### *3.1. The modified Gamma kernel estimator*

To make our discussion in the study of $M/G/1$ queuing system with optional second service using the non-parametric kernel method, we suppose that the variable $T$ is an exponential inter-arrival time with rate $\lambda > 0$, service $S$ and re-service $\widetilde{S}$ times are independent and drawn from univariate distributions having the pdfs $f_1$ and $f_2$, with means $\mu_1$, $\mu_2$ and variances $\delta_1$, $\delta_2$ respectively. Because the support of $S$ and $\widetilde{S}$ is $[0,\infty)$, then we employ the asymmetric (modified) Gamma kernel to estimate the pdfs of $S$ and $\widetilde{S}$. Let $K_{\text{MG}}(y;x,h)$ be a MG kernel that depends on the design point $x$ and the smoothing parameter $h$, [7, 9] for more details. The MG kernel estimator of $f_1(s)$ for service time $S$ is given as

$$\begin{aligned}\widehat{f}_1(s) &= \frac{1}{n_{s_1}}\sum_{i=1}^{n_{s_1}} K_{\text{MG}}(S_i; s, h)\\ &= \frac{1}{n_{s_1}}\sum_{i=1}^{n_{s_1}} \frac{S_i^{\rho_h(s)-1}\exp\{-S_i/h\}}{h^{\rho_h(s)}\Gamma\{\rho_h(s)\}}, s \geq 0. \end{aligned} \tag{1}$$

For the re-service time $\widetilde{S}$, the kernel estimator of $f_2(\widetilde{s})$ based on the MG kernel is also defined as

$$\begin{aligned}\widehat{f}_2(\widetilde{s}) &= \frac{1}{n_{s_2}}\sum_{i=1}^{n_{s_2}} K_{\text{MG}}(\widetilde{S}_i; \widetilde{s}, h)\\ &= \frac{1}{n_{s_2}}\sum_{i=1}^{n_{s_2}} \frac{\widetilde{S}_i^{\rho_h(\widetilde{s})-1}\exp\{-\widetilde{S}_i/h\}}{h^{\rho_h(\widetilde{s})}\Gamma\{\rho_h(\widetilde{s})\}}, \widetilde{s} \geq 0. \end{aligned} \tag{2}$$

Where $\{S_i\}_{i=1}^{n_{s_1}}$ and $\{\widetilde{S}_i\}_{i=1}^{n_{s_2}}$ are drawn from the densities $f_1$ and $f_2$, respectively, and

$$\rho_h(x) = \begin{cases} x/h & \text{for } x \geq 2h;\\ \frac{1}{4}(x/h)^2 + 1 & \text{for } x \in [0, 2h). \end{cases}$$

**Bandwidth selection.** The performance of the MG kernel estimators given by (1) and (2) depend on the bandwidth $h$, which controls the smoothness of these estimators. The optimal bandwidths which minimize the mean integrated squared error (MISE) of the estimator $\widehat{f}_j$ ($j = 1$ for service and $j = 2$ for re-service)

$$\begin{aligned}\text{MISE}(h) &= \mathbb{E}\left[\int_0^\infty \{\widehat{f}_j(x) - f_j(x)\}^2 dx\right]\\ &= \frac{1}{4}h^2\int_0^\infty \{x f_j''(x)\}^2 dx + \frac{1}{2n\sqrt{h\pi}}\int_0^\infty \frac{f_j(x)}{\sqrt{x}}dx + o\left(\frac{1}{n\sqrt{h}} + h^2\right). \end{aligned}$$

are (see [7, 9])

$$h_j^* = \left\{\frac{\frac{1}{2\sqrt{\pi}}\int_0^\infty x^{-1/2} f_j(x)dx}{\int_0^\infty x f_j''(x)\}^2 dx}\right\}^{2/5} n_j^{-2/5}, \quad j = 1, 2. \tag{3}$$

Where $\widehat{f}_j$ for $j = 1, 2$ are given by the eqs (1) and (2). The optimal bandwidth (3) depends on the unknown quantities $f_j$ and $f_j''$ for service time $S$, and re-service time $\widetilde{S}$. In order to overcome this problem in practice, we propose to use the unbiased cross validation (UCV) method. This approach has been developed by several authors, see for example [17] and [6].

The UCV technique can be adapted for MG kernel estimator. For a given MG kernel with target $x > 0$ and bandwidth $h > 0$, the optimal bandwidth $h_{\text{ucv}}$ is obtained by

$$h_{\text{ucv}} = \arg\min_h \text{UCV}(h)$$

$$\begin{aligned} \text{UCV}(h) &= \int_0^\infty \{\widehat{f}_j(x)\}^2 dx - \frac{2}{n_{s_j}} \sum_{i=1}^{n_{s_j}} \widehat{f}_j^{(-i)}(X_i) && (4) \\ &= \int_0^\infty \left\{ \frac{1}{n_{s_j}} \sum_{i=1}^{n_{s_j}} K_{\text{MG}}(X_i; x, h) \right\}^2 dx - \frac{2}{n_{s_j}(n_{s_j}-1)} \sum_{i=1}^{n_{s_j}} \sum_{k \neq i}^{n_{s_j}} K_{\text{MG}}(X_k; X_i, h). && (5) \end{aligned}$$

Where $\{X_i\}_{i=1}^{n_{s_j}}$ is drawn from unknown density $f_1$ for service or $f_2$ for re-service. Note that we can easily show that the UCV$(h)$ is an unbiased estimator of MISE$(h)$.

### *3.2. Performance mesures approximation based on modified Gamma kernel method*

In this section, we provide estimators for several performance measures of the system based on the modified Gamma kernel method.

- The estimators of the Mean Busy Period $\mathbb{E}(busy\, period)$ and the Probability of an Idle Period $P(idle\ period)$ of the system are given respectively, by

$$\frac{\hat{\mu}_1 + \hat{p}^2 \hat{\mu}_2}{1 - \hat{\lambda}\hat{\mu}_1}, \quad \frac{1 - \hat{\lambda}\hat{\mu}_1}{1 + \hat{p}^2 \hat{\lambda}\hat{\mu}_2}.$$

  Where $\hat{\mu}_1 = \int_0^\infty s\hat{f}_1(s)ds$, $\hat{\mu}_2 = \int_0^\infty \tilde{s}\hat{f}_2(\tilde{s})d\tilde{s}$ and $\hat{p} = \sum_{i=1}^{n_p} u_i/n_p$ where $u = \{u_i\}_{i=1}^{n_p}$ is a set of indicator variables such that $u_i = 1$ if customer $i$ requires re-service, and $u_i = 0$ otherwise, see [14] and [15].
- The Mean System Size estimators of $\mathbb{E}(X_n)$ and $\mathbb{E}(Y_n)$ are given by

$$\hat{\rho}_1 + \frac{\hat{\lambda}^2\hat{\delta}_1 + \hat{\rho}_1{}^2}{2(1-\hat{\rho}_1)} + \frac{\hat{p}[\hat{\lambda}^2\hat{\delta}_2 + \hat{\rho}_2^2 + \hat{p}\hat{\rho}_2^2(\frac{\hat{\lambda}^2\hat{\delta}_1 + \hat{\rho}_1(2-\hat{\rho}_1)}{1-\hat{\rho}_1})]}{2(\hat{p}\hat{\rho}_2 + (1-\hat{\rho}_1)\Psi(1-\hat{p}+\hat{p}B_2^*(\hat{\lambda})))},$$

  and

$$\frac{\hat{p}(2(1-\hat{\rho}_1)^2 + \hat{\lambda}^2\hat{\delta}_1 + \hat{\rho}_1(1-\hat{\rho}_1))}{2(\hat{p}\hat{\rho}_2 + (1-\hat{\rho}_1)\Psi(1-\hat{p}+\hat{p}B_2^*(\hat{\lambda})))}.$$

  Where,
  $B_1^*(v) = \int_0^\infty e^{-vs}\hat{f}_1(s)ds$, $B_2^*(v) = \int_0^\infty e^{-v\tilde{s}}\hat{f}_2(\tilde{s})d\tilde{s}$ are the LST estimators, and $\hat{\delta}_1 = \int_0^\infty s^2\hat{f}_1(s)ds - \hat{\mu}_1^2$, $\hat{\delta}_2 = \int_0^\infty \tilde{s}^2\hat{f}_2(\tilde{s})d\tilde{s} - \hat{\mu}_2^2$ are the service and re-service variances estimators, and $\widehat{\rho}_1 = \hat{\lambda}\hat{\mu}_1$ and $\widehat{\rho}_2 = \hat{\lambda}\hat{\mu}_2$, with $\hat{\lambda} = n_t/\sum_{i=1}^{n} T_i$ is the estimator of $\lambda$ obtained from exponential observed $n_t$ inter-arrival times $\{T_i\}_{i=1}^{n_t}$.

The general steps of the MG kernel estimator algorithm used for performance evaluation of M/G/1 system with optional second service are presented.

**Algorithm 1:** MG Kernel estimator algorithm for performance evaluation

**Data:** Collect the observed or simulated data, $\{S_i\}_{i=1}^{n_{s_1}}$ for service time and $\{\widetilde{S}_i\}_{i=1}^{n_{s_2}}$ for re-service time

**Bandwidth Selection**:

Choose the asymmetric MG kernel $K_{\text{MG}}(S; s, h) = \frac{S^{\rho_h(s)-1}\exp\{-S/h\}}{h^{\rho_h(s)}\Gamma\{\rho_h(s)\}}$, $s \geq 0$;

**for** $j = 1$ **to** $2$ **do**

  **for** $i = 1$ **to** $n_{s_j}$ **do**

    Compute the $UCV_j(h)$ criterion based on the samples $\{S_i\}_{i=1}^{n_{s_1}}$ for service time and $\{\widetilde{S}_i\}_{i=1}^{n_{s_2}}$ for re-service time

  **end**

  Minimize $UCV_j(h)$ and obtain the bandwidths $h_{ucv}^{s_1}$ for service and $h_{ucv}^{s_2}$ for re-service time

**end**

**Computation of the MG kernel estimators**

**for** $j = 1$ **to** $2$ **do**

  **for** $i = 1$ **to** $n$ **do**

    Compute the MG kernel estimators $\widehat{f}_j(s)$ based on the samples $\{S_i\}_{i=1}^{n_{s_1}}$ for service time and $\{\widetilde{S}_i\}_{i=1}^{n_{s_2}}$ for re-service time, by using the $UCV$ bandwidths

  **end**

  Compute the integrals $\hat{\mu}_j = \int_0^\infty s\hat{f}_j(s)ds$, $\hat{\delta}_j = \int_0^\infty s^2\hat{f}_j(s)ds - \hat{\mu}_j^2$, $B_j^*(v) = \int_0^\infty e^{-vs}\hat{f}_j(s)ds$.

**end**

**Result:** Estimated performance $\mathbb{E}(\textit{busy period})$, $P(\textit{idle period})$, $\mathbb{E}(X_n)$, $\mathbb{E}(Y_n)$ using $\hat{\mu}_j$, $\hat{\delta}_j$, $B_j^*(v)$.

## 4. Simulation Study

This section analyzes the performance of the proposed non-parametric model based on MG kernel using two simulated scenarios of the M/G/1 queuing system with an optional second service. These two scenarios have already been studied in [15] by using the mixture Gamma model combined with the Bayesian approach for parameters estimation. In the first scenario, service and re-service time distributions are represented as mixtures of Gamma distributions: two components for service times and three for re-service times. The second scenario explores a more intricate case, where the service time distribution is a mixture of two truncated Normal distributions, while the re-service time follows a Log-Normal distribution. Some important performance measures, such as the mean system size, mean busy period, and probability of an idle period, are computed. We compare the obtained performance of the estimators using the integrated squared error (ISE) criterion given by

$$\text{ISE} = \int_0^\infty \{\widehat{f}(x) - f(x)\}^2 dx,$$

where $\widehat{f}$ denotes an estimator that is obtained either via the MG kernel method or the Mixture Gamma approach of [15]. For both scenarios, $500$ replications are generated for each model, with sample sizes $n = 500$ and $n = 1000$. Note that all computations are given using the statistical R software. For comparison, we have used the `bmixture` R-package, that provides the Bayes estimates of mixture Gamma distributions; see [15] and [13].

### *4.1. The first scenario: mixture of Gamma*

We implemented the non-parametric approach described in Section 3 on the service and re-service dataset. The inter-arrival rate $\lambda$ is assumed to be known and equal to $0.26$, while the probability of re-service $p$ is also given and fixed at $0.3$.
We analyze a service data derived from a mixture of two Gamma distributions, as presented below.

$$f_1(x) = 0.6G(12, 3) + 0.4G(1, 2).$$

Additionally, for the re-service, the data are simulated from a mixture of three Gamma distributions, as shown below.

$$f_2(x) = 0.6G(100, \frac{100}{3}) + 0.3G(200, 50) + 0.1G(300, 60).$$

where the Gamma density function is given by:

$$G(x, \alpha, \beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}, x > 0.$$

Table 1. Performance measures results for $n = 500$ for the first scenario.

| Performance measures | $n = 500$ | | | | |
|---|---|---|---|---|---|
| | True value | Mixture Gamma with Bayes | | Non-Parametric Model | |
| | | Estimates | SD | Estimates | SD |
| $\mathbb{E}$(busy period) | 9.0525 | 8.8097 | 0.8798 | 9.5208 | 0.9777 |
| P(idle period) | 0.2982 | 0.3053 | 0.0202 | 0.2892 | 0.0209 |
| $\mathbb{E}(X_n)$ | 2.3597 | 2.4232 | 0.2972 | 2.4269 | 0.3172 |
| $\mathbb{E}(Y_n)$ | 0.2274 | 0.2077 | 0.01322 | 0.2018 | 0.0139 |
| $\mathbb{E}(\rho \mid data)$ | 0.9646 | 0.9350 | 0.0321 | 0.9540 | 0.0352 |
| $ISE$ (service) | - | 0.0091 | 0.0056 | 0.0043 | 0.0023 |
| $ISE$ (re-service) | - | 0.0663 | 0.0113 | 0.0704 | 0.0080 |

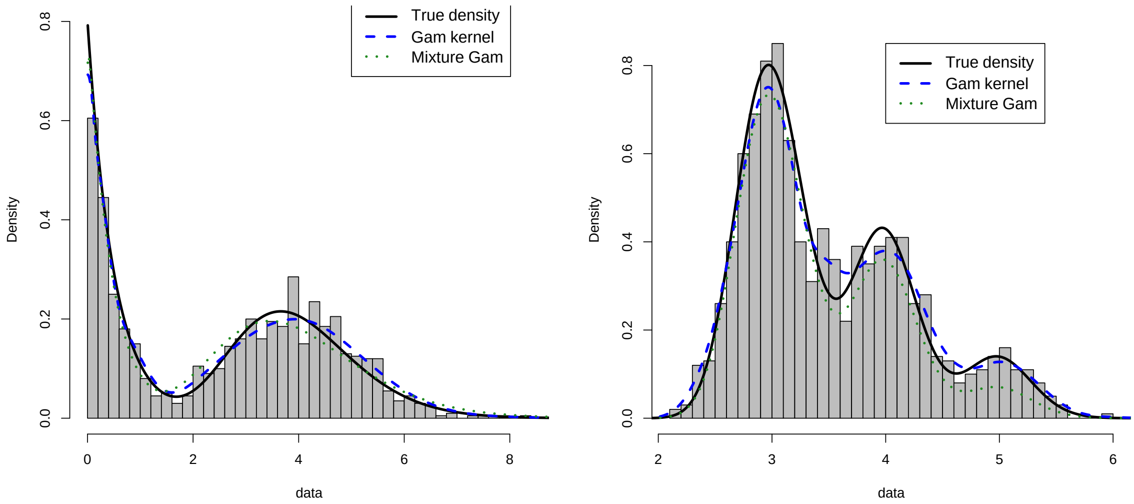


Figure 2. True PDF, mixture Gamma method and non-parametric MG kernel estimator for (left) service time and (right) re-service time from mixture Gamma data with $n = 1000$.

Table 2. Performance measures results for $n = 1000$ for the first scenario.

| Performance measures | $n = 1000$ | | | | |
|---|---|---|---|---|---|
| | True value | Mixture Gamma with Bayes | | Non-Parametric Model | |
| | | Estimates | SD | Estimates | SD |
| $\mathbb{E}$(busy period) | 9.0525 | 9.0259 | 0.6430 | 9.3814 | 0.6753 |
| P(idle period) | 0.2982 | 0.2995 | 0.0147 | 0.2915 | 0.0147 |
| $\mathbb{E}(X_n)$ | 2.3597 | 2.3819 | 0.1246 | 2.4128 | 0.1226 |
| $\mathbb{E}(Y_n)$ | 0.2274 | 0.2199 | 0.0017 | 0.2094 | 0.0018 |
| $\mathbb{E}(\rho \mid data)$ | 0.9646 | 0.9671 | 0.0218 | 0.9860 | 0.0243 |
| $ISE$ (service) | - | 0.0048 | 0.0014 | 0.0018 | 0.0009 |
| $ISE$ (re-service) | - | 0.0583 | 0.0119 | 0.0582 | 0.0054 |

The analysis of the results presented in the table 1 and table2 for $n = 500$ and $n = 1000$ respectively, allows us to evaluate the performance of two approaches the mixture Gamma model with Bayes estimates and the non-parametric Gamma kernel method by comparing their estimates to the true values of various system metrics. Both methods provide estimates for traffic intensity, system stability probability, mean number of customers in MQ and FQ, expected busy period and idle period probability that are relatively close to the true values.

We observed that the non-parametric kernel method performs better in terms of ISE for the service time when the sample size is $n = 500$ and $n = 1000$. However, for the re-service time, the mixture Gamma method with Bayes estimates outperforms the asymmetric MG kernel method when $n = 500$, and both methods yield very similar results when $n = 1000$. Moreover, the non-parametric model appears to yield more accurate estimates with smaller deviations from the true values.

Figure 2 presents the plots of the true density and the estimates (mixture Gamma with Bayes estimates and non-parametric MG kernel estimator) based on only one estimate from one simulated mixture Gamma models, with sample size $n = 1000$ for service (mixture of two Gamma) and re-service (mixture of three Gamma) times.

### *4.2. The second scenario: mixture of truncated Normal and Log-Normal*

As before, we assume that the inter-arrival rate ($\lambda = 0.28$) and the re-service probability ($\rho$= 0.45) are known. We use observations from a mixture of two truncated normal distributions on $(0, \infty)$.

$$f(x) = 0.4TN_{(0,\infty)}(1.4, 2.3) + 0.6TN_{(0,\infty)}(0.2, 0.3).$$

where normal density function which has been truncated on interval $(-\infty, 0)$, is given by:

$$TN_{(0,\infty)}(x, \mu, \sigma^2) = \frac{1}{\Phi(\mu/\sigma)\sqrt{2\pi\sigma^2}} \exp\left(\frac{-1}{2\sigma^2}(x - \mu^2)\right), x > 0.$$

Additionally, for the re-service, the data were simulated from a single Log-Normal distribution, as shown below.

$$f(x) = LN(1, 0.5)$$

where Log-Normal density function is given by:

$$LN(x, \mu, \sigma^2) = \frac{1}{x\sqrt{2\pi\sigma^2}} \exp\left(\frac{-1}{2\sigma^2}(\ln x - \mu^2)\right), x > 0$$

Table 3. Performance measures results for $n = 500$ for the second scenario.

| Performance measures | $n = 500$ | | | | |
|---|---|---|---|---|---|
| | True value | Mixture Gamma with Bayes | | Non-Parametric Model | |
| | | Estimates | SD | Estimates | SD |
| $\mathbb{E}$(busy period) | 5.7974 | 5.6978 | 0.0785 | 5.8612 | 0.0768 |
| P(idle period) | 0.3810 | 0.3488 | 0.0391 | 0.4123 | 0.0352 |
| $\mathbb{E}(X_n)$ | 1.661 | 1.4223 | 0.0635 | 1.5625 | 0.0662 |
| $\mathbb{E}(Y_n)$ | 0.227 | 0.2286 | 0.0029 | 0.2165 | 0.0031 |
| $\mathbb{E}(\rho \mid data)$ | 0.9830 | 0.9692 | 0.0144 | 0.9615 | 0.0150 |
| $ISE$ (service) | - | 0.0569 | 0.0176 | 0.0467 | 0.0034 |
| $ISE$ (re-service) | - | 0.0031 | 0.0021 | 0.0030 | 0.0021 |

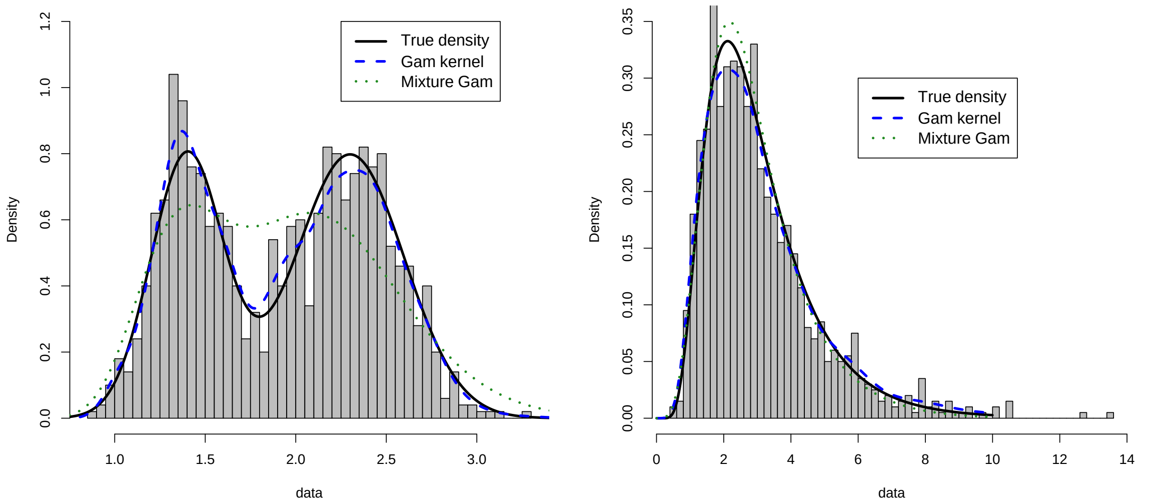


Figure 3. True PDF, mixture Gamma method and non-parametric MG kernel estimator for (left) service time and (right) re-service time from mixture of truncated Normal and Log-Normal data with $n = 1000$.

Table 4. Performance measures results for $n = 1000$ for the second scenario.

| Performance measures | $n = 1000$ | | | | |
|---|---|---|---|---|---|
| | True value | Mixture Gamma with Bayes | | Non-Parametric Model | |
| | | Estimates | SD | Estimates | SD |
| $\mathbb{E}$(busy period) | 5.7974 | 5.8393 | 0.0752 | 5.8402 | 0.0744 |
| P(idle period) | 0.3810 | 0.3652 | 0.0351 | 0.3598 | 0.0324 |
| $\mathbb{E}(X_n)$ | 1.6610 | 1.5428 | 0.0611 | 1.5680 | 0.0687 |
| $\mathbb{E}(Y_n)$ | 0.2270 | 0.2411 | 0.0026 | 0.2169 | 0.0028 |
| $\mathbb{E}(\rho \mid data)$ | 0.9830 | 0.9754 | 0.0091 | 0.9741 | 0.0102 |
| $ISE$ (service) | - | 0.0578 | 0.0125 | 0.0331 | 0.0024 |
| $ISE$ (re-service) | - | 0.0029 | 0.0020 | 0.0019 | 0.0012 |

The results presented in Table 3 and Table 4 for $n = 500$ and $n = 1000$, respectively, enables us to assess the performance of two approaches: the mixture Gamma with Bayes estimates and the non-parametric method based on the MG kernel. This evaluation is based on a comparison of their estimates with the true values of various system metrics. Both methods provide estimates for traffic intensity, system stability probability, the mean number of customers in MQ and FQ, the expected busy period, and the idle period probability, all of which remain relatively close to the true values. It is observed that the ISE for service and re-service in the non-parametric method is better than that obtained with the mixture Gamma method based on the Bayesian estimates. However, the parametric model tends to produce more accurate estimates, with smaller deviations from the true values.

Figure 3 shows the plots of the true density and the estimates (mixture Gamma with Bayes estimates and non-parametric MG kernel estimator) based on only one estimate from one simulated of truncated Normal and Log-Normal, with sample size $n = 1000$ for service (mixture of two truncated Normal) and re-service (Log-normal) times.

## 5. Conclusion

In this paper we show the interest of using the non-parametric method in the study of the M/G/1 queuing system with optional second service. This study is related to the work of [15] and provides a comparison between our method, non-parametric approach using MG kernel method, and mixture Gamma method with Bayes estimates. In our study, we employed the Gamma kernel method to estimate the service and re-service time densities when the underlying distribution $G$ is unknown. We considered two different simulation scenarios. In the first scenario, service and re-service times distributions are represented as mixtures of Gamma distributions: two components for service times and three for re-service times. The second scenario explores a more intricate case, where the service time distribution is a mixture of two truncated Normal distributions, while the re-service time follows a Log-Normal distribution.
Some important performance measures, such as the mean system size, mean busy period, and probability of an idle period, have been predicted.

Our simulation studies show that the proposed non-parametric method based on modified Gamma kernel for estimating the general distribution produces results that are favorably comparable to those obtained with the mixture Gamma model in terms of the ISE. Moreover, the obtained results demonstrate the efficiency of the proposed approach which can be considered as a good alternative to the existing classical methods.

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