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On a Computational Method for Two Asymmetric Parallel Queues

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Abstract: This paper addresses a well-known problem in queuing theory: the asymmetric shortest queue problem. The system consists of two parallel queues fed by a common Poisson arrival stream with rate λ. Upon arrival, each customer joins the shortest queue and remains there until being served. If the two queues have the same length, the arriving customer chooses the queue 1 with probability α and the queue 2 with probability $1-\alpha$. Service times are exponentially distributed, with rate μ_1 for the queue 1 and rate μ_2 for the queue 2. No jockeying is permitted between the two queues. An easy and efficient method is presented for computing the steady state solution of the system.

Keywords: Nonsymmetric shortest queue, numerical solution, simplex method.

1. Introduction

We consider two queues in parallel with a JSQ (joining the shortest queue) policy. The customers arrive to the system in accordance with a Poisson process of rate λ. Upon his arrival to the system, the customer joins the shortest queue and stays there until being served. If the two queues have the same length, the arriving customer chooses the queue 1 with probability α and the queue 2 with probability $1-\alpha$. Service times are exponentially distributed,with rate μ_1 for the queue 1 and rate μ_2 for the queue 2. No jockeying is allowed between the two queues.The aim of this paper is to provide an algorithm for computing the steady state probabilities for the system as is done in [6], [12] and [15], due to the structure of the infinitesimal generator. The shortest queue problem was initiated by Haight [7]. A successful model for the symmetric case $\left(\mu_1 = \mu_2 \text{ and } \alpha = \frac{1}{2}\right)$ was given when the jockeying between the two queues is possible. Kingman [9] obtained some asymptotic results for the joint steady state distribution of the number of customers in the two queues. Flatto and Mckean [5] used generating function techniques to give some limiting properties for the steady state probability. Halfin [8] performed a linear programming method for the problem by giving a lower and an upper bound for the probability distribution of the total number of customers in the system. Zhao and Grassmann [17] used the Flatto and McKean [5] results to develop a numerical solution. Adan, Wessels and Zijm [1] showed that the steady state distribution of the queue length is a mixture of product form distributions. Wang and Locker

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[14] presented a model where the state space of the related Markov process was truncated into banded arrays, they derive the probability distribution of queue length and the customer sojourn time. For queues with an infinite number of servers, efficient methods have been proposed in [10] and [16]. Blanc [3] introduced the power-series algorithm to evaluate the queue length distribution for a multiserver system. For the asymmetric case; Adan, Wessels and Zijm [2] adapted the product form representation for the symmetric case [1] and gave an efficient algorithm for the computation of the steady state probabilities by using the compensation approach. The method was satisfactory for the obtained results. Some performance measures are obtained in a model with jockeying [18]. A complete analytical solution was available as of 1998 , Cohen [4] expressed the bivariate generating function over four generating functions. The purpose of the present paper is to generalize and improve the presentation of the numerical method introduced in Lardjane and Messaci [10] and [11] to the asymmetric problem. The computation algorithm is based on the convex method initiated in [13]. We provide a numerical algorithm for the computation of the steady state probabilities. The method works efficiently in the uniform way for all situations (unbalanced systems, high traffic intensity).

2. Steady State Analysis of the System

Let X_t and Y_t be the number of customers in the queue 1 and 2 respectively at time t. The process $(X_t, Y_t)_t$ is a recurrent positive Markov process if $\lambda < \mu_1 + \mu_2$. We note E and $Q = (q(e, e'))_{(e,e')\in E\times E}$ the related state space and infinitesimal generator matrix. Let $p(i,j) = \lim\limits_{t\to\infty} P(X_t = i, Y_t = j)$ the steady state solution for the above process. We show further that $p(i,j)$ for $i + j = n$ is a function of the probabilities $(\pi_k)_{0\le k\le n}$ with $\pi_k = \sum\limits_{i+j=k} p(i,j)$. So, the computation of the steady state probabilities $p(i,j)$ for all (i,j) is reduced to the computation of the probabilities π_k. These last probabilities are computed by using a numerical algorithm based on the simplex method.

2.1. Notations

For $n \ge 1$ we define the $(n+1)\times 1$ vector $\boldsymbol{X}_n$ and the $(n+1)\times(n+1)$ matrice $\boldsymbol{A}_n$ by:

$$\boldsymbol{X}_n = (p(n,0), p(n-1,1), ..., p(1,n-1), p(0,n))^t, \boldsymbol{A}_1 = \begin{pmatrix} \mu_1 & \mu_2 \\ 1 & 1 \end{pmatrix}$$

and for $n \ge 2$:

$$\boldsymbol{A}_n = \begin{pmatrix} \mu_1 & \mu_2 & 0 & . & . & . & . & 0 \\ 0 & \mu_1 & \mu_2 & 0 & . & . & . & 0 \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ 0 & . & . & . & 0 & \mu_1 & \mu_2 & 0 \\ 0 & . & . & . & . & 0 & \mu_1 & \mu_2 \\ 1 & 1 & . & . & . & . & 1 & 1 \end{pmatrix}$$

We note that $\boldsymbol{A}_n$ is an inversible matrice for $\mu_1 \neq \mu_2$. We put $\pi_k = \sum\limits_{i+j=k} p(i,j)$ and define the $(n+1) \times 1$ vectors $\mathbf{B}_n$:

$$\boldsymbol{B}_1 = \begin{pmatrix} \lambda p\,(0,0) \\ \pi_1 \end{pmatrix}, \mathbf{B}_2 = \begin{pmatrix} (\lambda + \mu_1)\, p(1,0) - \lambda\alpha p\,(0,0) \\ (\lambda + \mu_2)\, p(0,1) - \lambda(1-\alpha) p\,(0,0) \\ \pi_2 \end{pmatrix},$$

$$\boldsymbol{B}_3 = \begin{pmatrix} (\lambda + \mu_1)\, p(2,0) \\ (\lambda + \mu_1 + \mu_2)\, p(1,1) - \lambda\pi_1 \\ (\lambda + \mu_2)\, p(0,2) \\ \pi_3 \end{pmatrix}$$

and for $k \geq 2$:

$$\boldsymbol{B}_{2k} = \begin{pmatrix} (\lambda + \mu_1)\, p(2k-1,0) \\ (\lambda + \mu_1 + \mu_2)\, p(2k-2,1) - \lambda p(2k-2,0) \\ . \\ . \\ (\lambda + \mu_1 + \mu_2)\, p(k,k-1) - \lambda\alpha p(k-1,k-1) - \lambda p\,(k,k-2) \\ (\lambda + \mu_1 + \mu_2)\, p(k-1,k) - \lambda\,(1-\alpha)\, p(k-1,k-1) - \lambda p\,(k-2,k) \\ . \\ . \\ (\lambda + \mu_1 + \mu_2)\, p(1,2k-2) - \lambda p(0,2k-2) \\ (\lambda + \mu_2)\, p(0,2k-1) \\ \pi_{2k} \end{pmatrix}$$

$$\boldsymbol{B}_{2k+1} = \begin{pmatrix} (\lambda + \mu_1)\, p(2k,0) \\ (\lambda + \mu_1 + \mu_2)\, p(2k-1,1) - \lambda p(2k-1,0) \\ (\lambda + \mu_1 + \mu_2)\, p(2k-2,2) - \lambda p(2k-2,1) \\ . \\ . \\ (\lambda + \mu_1 + \mu_2)\, p(k,k) - \lambda\,(p(k-1,k) - \lambda p(k,k-1)) \\ . \\ . \\ (\lambda + \mu_1 + \mu_2)\, p(2,2k-2) - \lambda p(1,2k-2) \\ (\lambda + \mu_1 + \mu_2)\, p(1,2k-1) - \lambda p(0,2k-1) \\ (\lambda + \mu_2)\, p(0,2k) \\ \pi_{2k+1} \end{pmatrix}$$

2.2. Equilibrium equations

Theorem 1. *From the steady state system of equilibrium equations we get:*

$$\boldsymbol{A}_n \boldsymbol{X}_n = \boldsymbol{B}_n \qquad \textit{for} \quad n \geq 1$$

Proof. The first balance equation is:

$$\lambda p\left(0,0\right)=\mu_1 p(1,0)+\mu_2 p(0,1)\,. \tag{1}$$

So, we get $\boldsymbol{A}_1\boldsymbol{X}_1=\boldsymbol{B}_1$ where:

$\boldsymbol{A}_1=\begin{pmatrix}\mu_1 & \mu_2\\ 1 & 1\end{pmatrix}$, $\boldsymbol{X}_1=\begin{pmatrix}p(1,0)\\ p(0,1)\end{pmatrix}$ and $\boldsymbol{B}_1=\begin{pmatrix}\lambda p\left(0,0\right)\\ \pi_1\end{pmatrix}$.

The next balance equations are:

$$\begin{aligned}\left(\lambda+\mu_1\right)p(1,0) &= \lambda\alpha p(0,0)+\mu_1 p(2,0)+\mu_2 p(1,1), &(2)\\ \left(\lambda+\mu_2\right)p(0,1) &= \lambda\left(1-\alpha\right)p(0,0)+\mu_2 p(0,2)+\mu_1 p(1,1) &(3)\end{aligned}$$

This leads to the form $\boldsymbol{A}_2\boldsymbol{X}_2=\boldsymbol{B}_2$ where:

$\boldsymbol{A}_2=\begin{pmatrix}\mu_1 & \mu_2 & 0\\ 0 & \mu_1 & \mu_2\\ 1 & 1 & 1\end{pmatrix}$, $\boldsymbol{X}_2=\begin{pmatrix}p(2,0)\\ p(1,1)\\ p(0,2)\end{pmatrix}$

and

$\boldsymbol{B}_2=\begin{pmatrix}\left(\lambda+\mu_1\right)p(1,0)-\lambda\alpha p\left(0,0\right)\\ \left(\lambda+\mu_2\right)p(0,1)-\lambda\left(1-\alpha\right)p\left(0,0\right)\\ \pi_2\end{pmatrix}$

We also have:

$$\begin{aligned}\left(\lambda+\mu_1\right)p(2,0) &= \mu_1 p(3,0)+\mu_2 p(2,1), &(4)\\ \left(\lambda+\mu_1+\mu_2\right)p(1,1) &= \mu_1 p(2,1)+\mu_2 p(1,2)+\lambda p(1,0)+\lambda p(0,1), &(5)\\ \left(\lambda+\mu_2\right)p(0,2) &= \mu_2 p(0,3)+\mu_1 p(1,2). &(6)\end{aligned}$$

$$\begin{pmatrix}\mu_1 & \mu_2 & 0 & 0\\ 0 & \mu_1 & \mu_2 & 0\\ 0 & 0 & \mu_1 & \mu_2\\ 1 & 1 & 1 & 1\end{pmatrix}\begin{pmatrix}p(3,0)\\ p(2,1)\\ p(1,2)\\ p(0,3)\end{pmatrix}=\begin{pmatrix}\left(\lambda+\mu_1\right)p(2,0)\\ \left(\lambda+\mu_1+\mu_2\right)p(1,1)-\lambda p(1,0)-\lambda p(0,1)\\ \left(\lambda+\mu_2\right)p(0,2)\\ \pi_3\end{pmatrix}$$

For all $n\geq 4$, as showed in the transition diagram in figure 1, we see that from the system of balance equations the probabilities $\{p(i,j)/\quad i+j=n\}$ are in terms of the probabilities $\{p(i,j)/\quad i+j=n-1\}$ and $\{p(i,j)/\quad i+j=n-2\}$.

So for $n=2k,\quad k\geq 2$, the system of balance equations takes the form:

$$\left(\lambda+\mu_1\right)p(2k-1,0)=\mu_1 p(2k,0)+\mu_2 p(2k-1,1) \tag{7}$$

for $0<i<k-1$:

$$\begin{aligned}\left(\lambda+\mu_1+\mu_2\right)p(2k-1-i,i) &= \mu_1 p(2k-i,i)+\mu_2 p(2k-1-i,i+1)\\ &\quad +\lambda p(2k-1-i,i-1) &(8)\\ \left(\lambda+\mu_1+\mu_2\right)p(k-1,k) &= \mu_1 p(k,k)+\mu_2 p(k-1,k+1)\end{aligned}$$

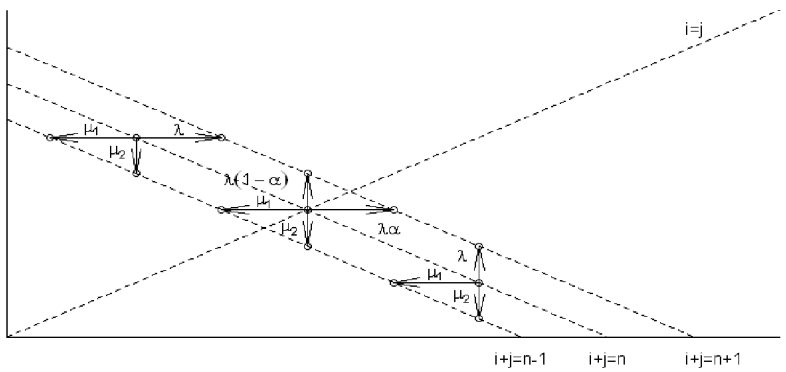

Figure 1. The transition diagram

$$+\lambda(1-\alpha)p(k-1,k-1)+\lambda p(k-2,k) \tag{9}$$

$$(\lambda+\mu_1+\mu_2)p(k,k-1)=\mu_2 p(k,k)+\mu_1 p(k+1,k-1)$$
$$+\lambda\alpha p(k-1,k-1)+\lambda p(k,k-2) \tag{10}$$

and for $k+1<i<2k-1$:

$$(\lambda+\mu_1+\mu_2)p(2k-1-i,i)=\mu_1 p(2k-i,i)+\mu_2 p(2k-1-i,i+1)$$
$$+\lambda p(2k-1-i,i-1) \tag{11}$$

$$(\lambda+\mu_2)p(0,2k-1)=\mu_1 p(1,2k-1)+\mu_2 p(0,2k) \tag{12}$$

The combination of the above balance equations leads to the matrice form of the proposition. So, we do for the case $n=2k+1$ for $k\geq 2$.

Proposition 2. *The m^{th} $(1\leq m\leq n+1)$ component of the vector $\boldsymbol{X}_n$ is of the form:*

$$\left(\sum_{l=1}^{n}\alpha_{l,m}^{(n)}x_l+\alpha_{0,m}^{(n)}\right)x_0 \qquad \alpha_{l,m}^{(n)}\in\mathbb{R} \tag{13}$$

where $x_l.x_0=\pi_l$ for $l\geq 1$, $p(0,0)=\pi_0=x_0>0$ and $\alpha_{l,m}^{(n)}$ are easily obtained over the matrices $\mathbf{A}_n$ and $\boldsymbol{B}_n$.

Proof. We use a recurrence argument. For $n=1$, we get from theorem 1

$$\boldsymbol{X}_1=\begin{pmatrix}p(1,0)\\p(0,1)\end{pmatrix}=\begin{pmatrix}\frac{1}{\mu_1-\mu_2} & -\frac{\mu_2}{\mu_1-\mu_2}\\ -\frac{1}{\mu_1-\mu_2} & \frac{\mu_1}{\mu_1-\mu_2}\end{pmatrix}\begin{pmatrix}\lambda x_0\\ x_1x_0\end{pmatrix}$$

$$=\begin{pmatrix}\lambda\frac{x_0}{\mu_1-\mu_2}-\mu_2x_0\frac{x_1}{\mu_1-\mu_2}\\ \mu_1x_0\frac{x_1}{\mu_1-\mu_2}-\lambda\frac{x_0}{\mu_1-\mu_2}\end{pmatrix}=\begin{pmatrix}x_0\frac{\lambda-\mu_2x_1}{\mu_1-\mu_2}\\ -x_0\frac{\lambda-\mu_1x_1}{\mu_1-\mu_2}\end{pmatrix}$$

$$= \begin{pmatrix} \left(\dfrac{-\mu_2}{\mu_1-\mu_2}x_1+\dfrac{\lambda}{\mu_1-\mu_2}\right)x_0 \\ \left(\dfrac{\mu_1}{\mu_1-\mu_2}x_1-\dfrac{\lambda}{\mu_1-\mu_2}\right)x_0 \end{pmatrix}$$

So: $\alpha_{1,1}^{(1)} = \dfrac{-\mu_2}{\mu_1-\mu_2}$, $\alpha_{0,1}^{(1)} = \dfrac{\lambda}{\mu_1-\mu_2}$, $\alpha_{1,2}^{(1)} = \dfrac{\mu_1}{\mu_1-\mu_2}$ and $\alpha_{0,2}^{(1)} = -\dfrac{\lambda}{\mu_1-\mu_2}$.
We then get the desired form for the components of the vector $\boldsymbol{X}_1$.We assume that the components of $\boldsymbol{X}_k$ $(2 \leq k \leq n-1)$ are of the form of the proposition. Then, the components of the vector $\boldsymbol{X}_n = \boldsymbol{A}_n^{-1}\boldsymbol{B}_n$ ($\boldsymbol{B}_n$ is given in terms of the $\boldsymbol{X}_{n-2}$ and $\boldsymbol{X}_{n-1}$ components) are of the proposition form.

Remark 1. *The solution of the system of balance equations is generated over the set* $(x_l)_l$.

3. On the Construction of the Solution

3.1. Notations

In order to build up a solution for the system of steady state equations we introduce the following positive real numbers $r(i,j)$. We set $r(0,0) = y_0$, where y_0 is any stictly positive real number ($r(0,0)$ is a very first approximation of $p(0,0)$ if $0 < r(0,0) < 1$). Let the convex set $C_1 = \left\{ y_1 > 0 / \left(\frac{-\mu_2}{\mu_1-\mu_2}y_1 + \frac{\lambda}{\mu_1-\mu_2}\right) > 0 \text{ and } \left(\frac{\mu_1}{\mu_1-\mu_2}y_1 - \frac{\lambda}{\mu_1-\mu_2}\right) > 0 \right\}$. C_1 is a nonempty set, it contains at least the positive real number x_1 defined in proposition 2. We then put

$$\boldsymbol{Y}_1 = \begin{pmatrix} r(1,0) \\ r(0,1) \end{pmatrix} = \begin{pmatrix} \left(\dfrac{-\mu_2}{\mu_1-\mu_2}y_1+\dfrac{\lambda}{\mu_1-\mu_2}\right)y_0 \\ \left(\dfrac{\mu_1}{\mu_1-\mu_2}y_1-\dfrac{\lambda}{\mu_1-\mu_2}\right)y_0 \end{pmatrix}.$$

So, $r(0,1)$ and $r(1,0)$ are a positive solution for the equilibrium equation:

$$\lambda r(0,0) = \mu_1 r(1,0) + \mu_2 r(0,1). \tag{14}$$

For some $y_1 \in C_1$, let $C_2 = \left\{ y_1 > 0, y_2 > 0 / \left(\sum_{l=1}^{2} \alpha_{l,m}^{(2)} y_l + \alpha_{0,m}^{(2)}\right) > 0 \text{ for } m = 1,2,3 \right\}$. C_2 is an $\mathbb{R}^2$ convex set. We again put

$$\boldsymbol{Y}_2 = \begin{pmatrix} r(2,0) \\ r(1,1) \\ r(0,2) \end{pmatrix} = \begin{pmatrix} \left(\sum_{l=1}^{2} \alpha_{l,1}^{(2)} y_l + \alpha_{0,1}^{(2)}\right) y_0 \\ \left(\sum_{l=1}^{2} \alpha_{l,2}^{(2)} y_l + \alpha_{0,2}^{(2)}\right) y_0 \\ \left(\sum_{l=1}^{2} \alpha_{l,3}^{(2)} y_l + \alpha_{0,3}^{(2)}\right) y_0 \end{pmatrix}.$$

$r(i,j)$, for $i+j=2$, are then a positive solution for the equilibrium equations:

$$(\lambda+\mu_1)\, r(1,0) = \lambda\alpha r(0,0) + \mu_1 r(2,0) + \mu_2 r(1,1), \tag{15}$$

$$(\lambda+\mu_2)\,r(0,1) \;=\; \lambda\,(1-\alpha)\,r(0,0)+\mu_2 r(0,2)+\mu_1 r(1,1). \tag{16}$$

And so on, we define:
$C_n=\left\{y_1>0,y_2>0,..,y_n>0/\left(\sum_{l=1}^{n}\alpha_{l,m}^{(n)}y_l+\alpha_{0,m}^{(n)}\right)>0 \text{ for } m=1,..,n+1\right\}.$
We then put:

$$\boldsymbol{Y}_n=\begin{pmatrix} r(n,0)\\ r(n-1,1)\\ .\\ .\\ .\\ r(1,n-1).\\ r(0,n)\end{pmatrix}=\begin{pmatrix}\left(\sum_{l=1}^{n}\alpha_{l,1}^{(n)}y_l+\alpha_{0,1}^{(n)}\right)y_0\\ \left(\sum_{l=1}^{n}\alpha_{l,2}^{(n)}y_l+\alpha_{0,2}^{(n)}\right)y_0\\ .\\ .\\ .\\ \left(\sum_{l=1}^{n}\alpha_{l,n}^{(n)}y_l+\alpha_{0,n}^{(n)}\right)y_0\\ \left(\sum_{l=1}^{n}\alpha_{l,n+1}^{(n)}y_l+\alpha_{0,n+1}^{(n)}\right)y_0\end{pmatrix}$$

Remark 2. *The positive real numbers $r\,(i,j)$ solve the same equilibrium equations as the $p\,(i,j)$. So, we can set $\boldsymbol{A}_n\boldsymbol{Y}_n=\boldsymbol{D}_n$. $\boldsymbol{D}_n$ has exactly the same form as $\boldsymbol{B}_n$ where $r\,(i,j)$ replaces $p\,(i,j)$ and π_n replaced by $R_k=\sum_{i+j=k} r\,(i,j)=y_k y_0$ for $k\geq 1$. By construction, for $(y_1,y_2,.....,y_n)\in C_n$, the set $\{r\,(i,j)\,,i+j\leq n\}$ is a positive solution for the system of equilibrium equations $r(e)\sum_{e'\neq e}q(e,e')=\sum_{e'\neq e}r(e')q(e',e)$ where $(e)=(i,j);i+j\leq n-1$.*

Proposition 3. *Let S_n the set of infinite sequences defined as follows: $S_n=\left\{\left(y_i\right)_{i\geq 1}/\left(y_1,y_2,.....,y_n\right)\in C_n\right\}$. $\left(S_n\right)_n$ is then a decreasing sequence of sets ($S_n\subset S_{n-1}$) and the limit $\cap S_n$ is (up to a multiplicative factor) the set of positive real numbers $\left(x_k\right)_{k\geq 1}$ defined in proposition 2.*

Proof. The decreasing property of the sequence $\left(S_n\right)_n$ is obvious and obtained by the definition of S_n.The ergodic property of the process $\left(X_t,Y_t\right)_t$ ensure the existence and the unicity (up to a multiplicative factor) of the limit $\cap S_n$.

Remark 3. *For n large enough, the subset C_n (as a projection of S_n on $\mathbb{R}^n$) is almost reduced to a point of $\mathbb{R}^n$. It's components constitute a generator set for the steady state solution of the considered system of balance equations.To see how the convergence to the solution for the two first components is made, we illustrate geometrically (for $\lambda=1,\mu_1=1,\mu_2=2$) the behavior of C_2 and the projection of C_3 and C_4 on the plane. (Figure 2).*

3.2. Computation of $\left(x_l\right)_l$

We note first that for any fixed n $(x_1,.....x_n)\in C_n$. In other words each $(y_1,....,y_n)\in C_n$ is a candidate for $(x_1,.....x_n)$. So y_l is an approximation of x_l. We then start the computation by the determination of C_n. To do this, we use the simplex method to minimize or maximize $\sum_{l=1}^{n} y_l$ with the constraints $r\,(i,j)\geq 0$ for $i+j=n$ and $y_l\geq 0$. We then get a lower and an upper bound for x_l $\left(y_{l,\min}\leq x_l\leq y_{l,\max}\right)$. $y_{l,\min}$ (resp. $y_{l,\max}$) is the lowest (resp. highest) value of y_l in minimizing (resp. maximizing) $\sum_{i=1}^{n} y_l$ with the given constraints.

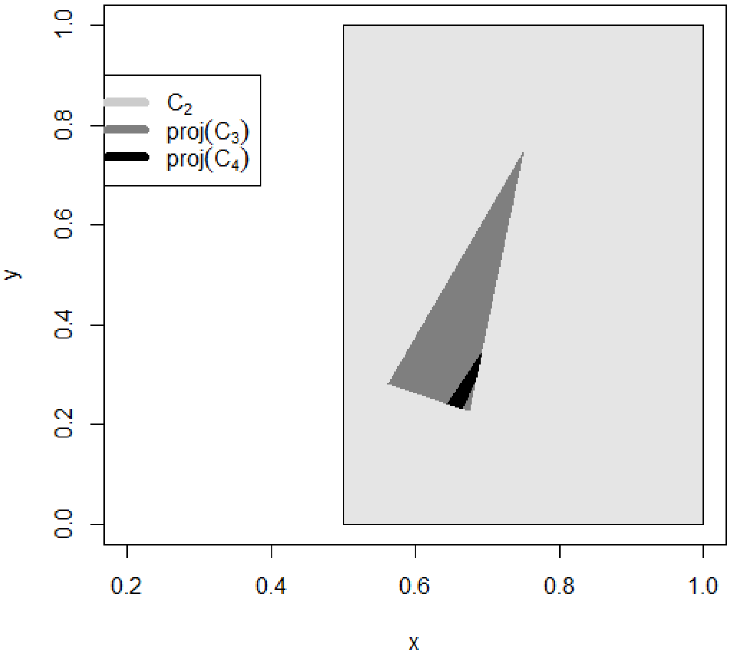

Figure 2. Evolution of the convex set solution

3.3. Algorithm description

Step 1: Set up the values of λ, μ_1, μ_2, α and a precision level ϵ.

Step 2: For N sufficiently large get the components of the vector $\boldsymbol{Y}_N$ which are of the form: $\left(\sum_{l=1}^{N} \alpha_{l,m}^{(N)} y_l + \alpha_{0,m}^{(N)}\right) y_0$

Step 3: Use the simplex method with objective function $\sum_{l=1}^{N} y_l$ and constraints:

$$\sum_{l=1}^{N} \alpha_{l,m}^{(N)} y_l + \alpha_{0,m}^{(N)} \geq 0 \quad 1 \leq m \leq N+1 \quad y_l \geq 0 \quad 1 \leq l \leq N.$$

Step 4: Get a lower and an upper bound for y_l denoted respectively $y_{l,\min}$ and $y_{l,\max}$ and check the precision $\underset{1 \leq l \leq N}{Max} \left(y_{l,\max} - y_{l,\min}\right) < \epsilon$. If yes put $x_l = \frac{1}{2}\left(y_{l,\min} + y_{l,\max}\right)$ and go to step 5, if no, return to step 2 and increase the value of N.

Step 5: Get y_0 over the normalisation equation $y_0 \left(1 + \sum_{l=1}^{N} x_l\right) = 1$, put $x_0 = y_0$ and $\pi_l = x_0 . x_l$.

Step 6: Build up the numerical solution over the system of equations $\boldsymbol{A}_k\boldsymbol{X}_k = \boldsymbol{B}_k$, for $k = 1, ..., n$.

3.4. Numerical results

The following tables show the computation accuracy of x_l ($y_{l,\min} \leq x_l \leq y_{l,\max}$) and the stationary probabilities obtained for the different values of the parameters.

Table 1. $\lambda = 1, \mu_1 = 5, \mu_2 = 10$ and $\alpha = 0.25$.

l	$y_{l,\min}$	$y_{l,\max}$
1	$1.2787268701\ 10^{-1}$	$1.2787268701\ 10^{-1}$
2	$8.7786116516\ 10^{-3}$	$8.7786116516\ 10^{-3}$
3	$5.8590092785\ 10^{-4}$	$5.8590092785\ 10^{-4}$
4	$3.9061841909\ 10^{-5}$	$3.9061841909\ 10^{-5}$
5	$2.6041277617\ 10^{-6}$	$2.6041277617\ 10^{-6}$
6	$1.7360853157\ 10^{-7}$	$1.7360853157\ 10^{-7}$
7	$1.1573902145\ 10^{-7}$	$1.1573902145\ 10^{-8}$
8	$7.7159347648\ 10^{-10}$	$7.7159347648\ 10^{-10}$.
9	$5.1439565099\ 10^{-11}$	$5.1439565099\ 10^{-11}$
10	$3.4293043399\ 10^{-12}$	$3.4293043399\ 10^{-12}$
11	$2.2862028933\ 10^{-13}$	$2.2862028933\ 10^{-13}$
12	$1.5241352622\ 10^{-14}$	$1.5241352622\ 10^{-14}$
13	$1.0160901748\ 10^{-15}$	$1.0160901748\ 10^{-15}$
14	$6.7739344987\ 10^{-17}$	$6.7739344987\ 10^{-17}$
15	$4.5159563324\ 10^{-18}$	$4.5159563324\ 10^{-18}$
16	$3.0106375549\ 10^{-19}$	$3.0106375549\ 10^{-19}$
17	$2.0070917033\ 10^{-20}$	$2.0070917033\ 10^{-20}$
18	$1.3380611355\ 10^{-21}$	$1.3380611355\ 10^{-21}$
19	$8.9204075703\ 10^{-23}$	$8.9204075704\ 10^{-23}$
20	$5.9469383799\ 10^{-24}$	$5.9469383811\ 10^{-24}$
21	$3.9646255844\ 10^{-25}$	$3.9646255964\ 10^{-25}$
22	$2.6430837008\ 10^{-26}$	$2.6430838202\ 10^{-26}$
23	$1.7620555801\ 10^{-27}$	$1.7620567722\ 10^{-27}$
24	$1.1747015832\ 10^{-28}$	$1.1747134837\ 10^{-28}$

Table 2. $\lambda = 60, \mu_1 = 1, \mu_2 = 100$ and $\alpha = 0.1$.

l	$y_{l,\min}$	$y_{l,\max}$
1	19.0485310693	19.0485310693
20	4.5863493910	4.5863493910
30	0.0251200856	0.0251200856
40	0.0001375058	0.0001375058
50	$7.526964\ 10^{-7}$	$7.526964\ 10^{-7}$
60	$4.12020298\ 10^{-9}$	$4.12020298\ 10^{-9}$
70	$2.255367\ 10^{-11}$	$2.255367\ 10^{-11}$
80	$1.234571\ 10^{-13}$	$1.234571\ 10^{-13}$
100	$3.699249\ 10^{-18}$	$3.703686\ 10^{-18}$

Table 3. $\lambda = 70, \mu_1 = 1, \mu_2 = 100$ and $\alpha = 0.1$.

l	$y_{l,\min}$	$y_{l,\max}$
1	23.279382	23.279382
10	25672.957703	25672.957703
20	1076.816070	1076.8160870
30	27.835448	27.835448
40	0.711977	0.711977
60	0.000465	0.000465
80	$3.044634\ 10^{-7}$	$3.044673\ 10^{-7}$
100	$1.990987\ 10^{-10}$	$2.030542\ 10^{-10}$

Table 4. Values of the $p(i,j)'s$ for $\epsilon = 10^{-23}, \lambda = 1, \mu_1 = 5, \mu_2 = 10$ and $\alpha = 0.25$.

$i\backslash j$	0	1	2	3	4	5	6
0	0.87929	$6.34\ 10^{-2}$	$1.65\ 10^{-4}$	$2.41\ 10^{-7}$	$3.42\ 10^{-10}$	$4.6\ 10^{-13}$	$1.4\ 10^{-39}$
1	$4.90\ 10^{-2}$	$7.30\ 10^{-3}$	$3.63\ 10^{-4}$	$5.29\ 10^{-7}$	$7.51\ 10^{-10}$	$1.0\ 10^{-12}$	$3.7\ 10^{-40}$
2	$2.52\ 10^{-4}$	$1.50\ 10^{-4}$	$3.33\ 10^{-5}$	$1.64\ 10^{-6}$	$2.33\ 10^{-9}$	$3.2\ 10^{-12}$	
3	$7.50\ 10^{-7}$	$4.49\ 10^{-7}$	$6.42\ 10^{-7}$	$1.48\ 10^{-7}$	$7.31\ 10^{-9}$	$1.0\ 10^{-11}$	
4	$2.18\ 10^{-9}$	$1.30\ 10^{-9}$	$1.86\ 10^{-9}$	$2.84\ 10^{-9}$	$6.58\ 10^{-10}$	$3.1\ 10^{-11}$	
5	$1.29\ 10^{-11}$	$3.93\ 10^{-12}$	$4.63\ 10^{-12}$	$6.99\ 10^{-12}$	$1.07\ 10^{-11}$		
6	$7.65\ 10^{-12}$	$7.24\ 10^{-13}$	$6.06\ 10^{-14}$	$3.78\ 10^{-15}$			
7	$7.73\ 10^{-12}$	$6.64\ 10^{-13}$	$4.15\ 10^{-14}$				
8	$7.95\ 10^{-12}$	$4.97\ 10^{-13}$					
9	$8.55\ 10^{-12}$						
10	$1.02\ 10^{-11}$						

Table 5. The steady state probabilities $p(i,j)$ for highly unbalanced queues; $\epsilon = 10^{-20}$, $\lambda = 1$, $\mu_1 = 1$, $\mu_2 = 100$ and $\alpha = 0.25$.

i/j	0	1	2	3	4	5
0	0.79016	0.00589	$0.149.10^{-6}$	$0.178.10^{-12}$	$0.172.10^{-18}$	$0.165.10^{-24}$
1	0.20137	0.00205	0.00002	$0.180.10^{-10}$	$0.174.10^{-16}$	$0.167.10^{-22}$
2	0.00052	0.00001	$0.250.10^{-6}$	$0.184.10^{-8}$	$0.177.10^{-14}$	$0.170.10^{-20}$
3	$0.625.10^{-7}$	$0.125.10^{-8}$	$0.650.10^{-9}$	$0.246.10^{-10}$	$0.181.10^{-12}$	
4	$0.608.10^{-11}$	$0.121.10^{-12}$	$0.632.10^{-13}$	$0.633.10^{-13}$		
5	$0.590.10^{-15}$	$0.118.10^{-16}$	$0.614.10^{-17}$			
6	$0.573.10^{-19}$	$0.114.10^{-20}$				
7	$0.556.10^{-23}$					

Table 6. The steady state probabilities $p(i,j)$ for highly unbalanced queues; $\epsilon = 10^{-13}$, $\lambda = 60$, $\mu_1 = 1$, $\mu_2 = 100$ and $\alpha = 0.1$.

i/j	0	1	2	3	4
0	0.0000720173	0.000029790	$1.6657.10^{-6}$	$4.271.10^{-8}$	$5.8306.10^{-10}$
1	0.00134203	0.000710899	0.000262246	$6.7761833.10^{-6}$	
2	0.01034207	0.005906328	0.003055594		
3	0.0388034	0.022872396			
4	0.079767769				

Table 7. The steady state probabilities $p(i,j)$ for highly unbalanced queues; $\epsilon = 10^{-7}$, $\lambda = 70$, $\mu_1 = 1$, $\mu_2 = 100$ and $\alpha = 0.1$.

i/j	0	1	2	3	4	5	6
0	$4.324.10^{-6}$	$2.04.10^{-6}$	$1.49.10^{-7}$	$5.3796.10^{-9}$	$1.1102.10^{-10}$	$1.45.10^{-12}$	$1.30.10^{-14}$
1	0.0000986	0.000059	0.000024	$9.034.10^{-7}$	$1.87.10^{-8}$	$2.47.10^{-10}$	
2	0.0010128	0.000664	0.000385	0.00015223	$3.17.10^{-6}$		
3	0.0055047	0.003731	0.0.002405	0.001391			
4	0.0117702	0.012200	0.008215				
5	0.0367969	0.025592					
6	0.053368						

Table 8. The steady state probabilities $p(i,j)$ for highly unbalanced queues; $\epsilon = 10^{-3}, \lambda = 80$, $\mu_1 = 1$, $\mu_2 = 100$ and $\alpha = 0.1$.

i/j	0	1	2	3	4
0	$1.006.10^{-7}$	$5.32.10^{-8}$	$4.86.10^{-9}$	$2.33.10^{-10}$	$6.77.10^{-12}$
1	$2.72.10^{-6}$	$1.84.10^{-6}$	$8.52.10^{-7}$	$4.13.10^{-8}$	
2	0.00035	0.0000262	0.000016		
3	0.000263	0.000201			
4	0.00122				

Table 9. The steady state probabilities $p(i,j)$ for highly unbalanced queues and heavy traffic; $\epsilon = 10^{-2}, \lambda = 90$, $\mu_1 = 1$, $\mu_2 = 100$ and $\alpha = 0.1$.

i/j	0	1	2	3	4	9	10	29	30
0	$3.4.10^{-10}$	2.10^{-10}	2.10^{-11}	1.10^{-12}	5.10^{-14}	5.10^{-23}	5.10^{-25}	6.10^{-53}	1.10^{-52}
1	$1.09.10^{-8}$	$8.1.10^{-9}$	4.10^{-9}	2.10^{-10}	9.10^{-12}				
2	1.10^{-7}	1.10^{-7}	9.10^{-8}	4.10^{-8}					
3	$1.6.10^{-6}$	$1.1.10^{-6}$	$1.1.10^{-6}$						
4	0.00001	$9.2.10^{-6}$							
9	0.0018	0.0016							
10	0.0030								
29	0.00121	0.00109							
30	0.00097								

4. Conclusion

An efficient method of computing the steady state probabilities for an asymmetric shortest queue problem is presented in this paper. The method gives a useful way for the solution of the problem described.The advances of the software technology(formal calculus and simplex algorithm) is the key tool for using such method and get results with a high precision. The method can be adapted for systems with more than two queues in parallel, some programming work is required.

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