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The Effects of a Pro Rata Rebate Warranty on the Discrete Age Replacement Policy with Salvage Value Consideration

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Abstract: Consider a system that should be operating over an indefinitely long operation cycle n (n = 1,2,...). Under the discrete time age-replacement policy, a system is replaced at the completion of cycle N (N = 1,2,...) or at failure, whichever occurs first. For the pro rata rebate warranty (PRRW), the customer will be refunded a proportion of the purchasing cost if the system fails within the warranty period. When the system is preventive replaced, a salvage value that proportional to its expected residual lifetime is gained. Cost models from the customer's perspective are developed for both warranted, and non-warranted systems. The corresponding optimal replacement age N^* is derived such that the long-run expected cost rate is minimized. Under the assumption of the discrete time increasing failure rate, the existence and uniqueness of the optimal N^* are shown, and the impacts of a PRRW on the optimal replacement policies are investigated analytically. Finally, a numerical example is demonstrated for the optimal policy illustration and verification. The observations from the technical analysis and numerical results provide valuable information for a buyer (user) to adjust their optimal preventive replacement policy when the system is operating in discrete time and under a PRRW.

Keywords: Age replacement, discrete failure distribution, increasing failure rate, long-run expected cost rate, pro rata rebate warranty.

1. Introduction

The classical age-replacement policy is proposed by Barlow and Proschan [2], in which an operating system is replaced at time of failure or at age T, whichever comes first. Another well-known preventive replacement policy proposed by Barlow and Hunter [1] is the classical periodic replacement policy (also called the block replacement policy), where an operating system is replaced by a new one at times kT $(k = 1, 2, 3, \cdots)$, and at failures. Afterwards, many authors have systematically studied and extended these two well-known replacement model, they become the most commonly used preventive

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maintenance (PM) policies in reliability theory. The aim of optimal PM policies is to provide optimum system reliability/availability and safety performance at the lowest possible maintenance cost.

In the modern marketplace, most products are sold with a warranty, thus, to incorporate various product warranties into the derivation of the optimal PM policy would be interesting and possibly useful. Jack and Schouten [16], Djamaludin and Murthy [15], Jung and Park [17], Chen and Chien [7], and Wu *et al.* [25] incorporate system warranty with various maintenance actions to investigate the performance of the optimal PM policies. Yeh *et al.* [26, 27], and Chien [8-11] analyzed the impacts of various warranties on the classical preventive replacement policies. However, all the warranty-replacement problems mentioned above are modeled under a continuous operating circumstance. In other words, since most of the warranty-replacement model are classified as continuous-time models, they will lose their validity in a discrete-time setting. In failure studies for airplane parts, the time to unit failure is often measured by the number of operation cycles to failure. In actual situations, jet fighter tires are replaced preventively after 4-14 flights, which may depend on the kind of use. In other cases, lifetimes are sometimes not recorded at the exact instant of failure but are collected statistically per day, per month, or per year. Therefore, in any case, it is interesting and possibly useful to consider discrete time processes. And after Nakagawa [23, 24] proposed a discrete time age-replacement policy, Chien [12, 13] and Chien and Zhang [14] incorporate the warranties into the replacement policy by considering the product is operating in a discrete time process: in Chien [12], the effects of a free-repair warranty (FRW) on the optimal discrete time periodic replacement policy is discussed; in Chien [13], the impacts of a renewing free-replacement warranty (RFRW) on the optimal discrete time age-replacement policy is investigated. Chien and Zhang [14] further analyzed a hybrid warranty policy for systems operating in discrete time.

A rebate warranty is one of the most common types of warranty policies. Under a rebate policy, the manufacturer (seller) refunds a customer (buyer) some proportion of the sales price if the product fails during the warranty period. Common examples of products sold under rebate policies include batteries and tires. In this paper, a pro rata rebate warranty (PRRW) is considered for deriving the optimal discrete time age-replacement policy, and the salvage value of an un-failed system that due to preventive replacement is also considered. From the customer’s perspective, a mathematical formulation for the long term expected cost rate is developed. Under the increasing failure rate (IFR) assumption, the existence and uniqueness of the optimal age for preventive replacement (i.e., the optimal number of operation cycles for preventive replacement) such that the long-run expected cost rate is minimized is shown. Furthermore, the optimal ages for preventive replacement, and the corresponding cost rates for systems with and without PRRW are compared analytically, and their structural properties are summarized.

The reminder of this paper is organized as follows. In Section 2, the model assumptions are described, and mathematical formulations for the expected cost rates are established. Based on the cost models, the optimal number of operation cycles for preventive replacement for both a warranted, and a non-warranted system are derived, and their structural properties are presented in Section 3. These optimal replacement policies and their corresponding expected cost rates are compared analytically in Section 4. In Section 5, a special case of the discrete failure distribution is considered as a numerical example, and sensitivity analysis of effectiveness of the model parameters on the optimal policies are performed. Finally, some comments are concluded in Section 6.

2. Mathematical Formulation

In this section, cost models from the customer's perspective are developed for both warranted, and non-warranted systems.

2.1. Preliminaries

Under the discrete time age-replacement policy, the system is replaced at the time when the Nth ($N = 1, 2, \cdots$) operation cycle is completed, or is replaced at failure, whichever occurs first. More precisely, when the system fails at operation $n\,(\le N)$, a failure replacement (corrective replacement) is performed with a downtime cost $C_d > 0$, and a purchasing cost $C_p > 0$. If the system passes through the cycle N and does not fail (i.e., the Nth operation cycle is completed successfully), then a preventive replacement is carried out. Because a preventive replacement is a planned PM action, only the cost C_p is incurred in this action. Therefore, under this model, the design variable is N.

Without considering warranty, various replacement policies in discrete time have been investigated by researchers [19, 20, 23, 24]. However, because the system, that preventive replaced at the completion of Nth operation cycle, is still operable, so the salvage value of an un-failed system should be considered in the cost model. It is reasonable to assume that the salvage value of a used (un-failed) system is proportional to its expected residual lifetime, thus, in this study, we define it as $v_s \cdot (n-N) \big| \{n > N\}$, which is similar to the definition used in Kaio and Osaki [18] and Chien [11]. On the other hand, under a PRRW, the customer is refunded a proportion of the sales prices C_p if the system fails within the warranty. Thus, the refund amount, $R(n)$, is a function of the failure time n, and we define it as

$$R(n) = \begin{cases} C_p\left(1 - \dfrac{n-1}{W}\right), & \text{for} \quad 1 \le n \le W, \\ 0, & \text{for} \quad n > W. \end{cases} \tag{1}$$

Then, by the similar method to that of Chien [12], the cost model for operating the system in discrete time, in a long run, can be established.

2.2. Cost model without warranty

Without warranty, any two successive replacements of the system form a renewal cycle of the failure process, Figure 1 illustrates this case.

Hence, the replacement cycle length (i.e., the renewal cycle length, denote by $T_0(N)$) is

$$T_0(N) = \begin{cases} n, & \text{if} \quad n \le N, \\ N, & \text{if} \quad n > N, \end{cases} \tag{2}$$

and the total cost incurred in a renewal cycle (denote by $C_0(N)$) is

$$C_0(N) = \begin{cases} C_d + C_p, & \text{if} \quad n \le N, \\ C_p - v_s \cdot (n - N), & \text{if} \quad n > N. \end{cases} \tag{3}$$

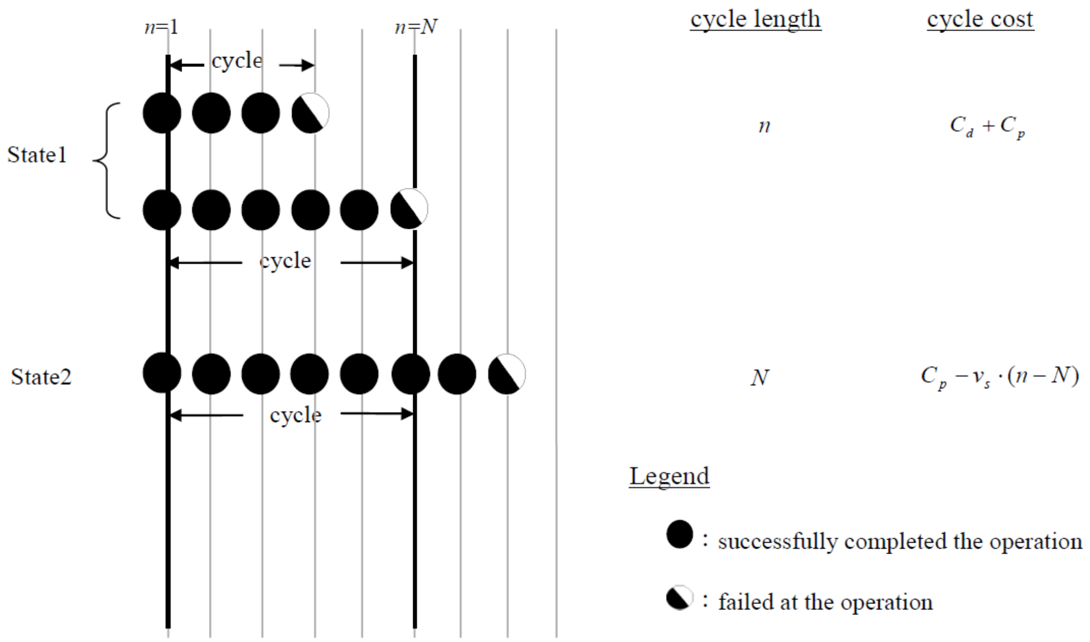

Figure 1. Possible replacements without warranty.

Thus, by (2) and (3), the long-run expected cost rate is

$$CR_0(N) = \frac{E[C_0(N)]}{E[T_0(N)]}$$

$$= \frac{(C_d + C_p) \cdot \sum_{n=1}^{N} p_n + \sum_{n=N+1}^{\infty} [C_p - v_s(n-N)] p_n}{\sum_{n=1}^{N} n \cdot p_n + N \cdot \sum_{n=N+1}^{\infty} p_n}$$

$$= \frac{C_p + C_d \sum_{n=1}^{N} p_n - v_s \sum_{m=N}^{\infty} \sum_{n=m+1}^{\infty} p_n}{\sum_{m=1}^{N} \sum_{n=m}^{\infty} p_n} . \tag{4}$$

2.3. Cost model with warranty

For a system purchased with the PRRW, the total cost incurred in a renewal cycle depends on whether a preventive replacement is scheduled within the warranty period or not. Thus, the cost model should be established for two cases: $N > W$ and $N \le W$.

Case 1. $N > W$

When the operation cycle for preventive replacement of a system is scheduled after the warranty expiration, then there exist three possible replacement states, as shown in Figure 2. First, if the system fails within the warranty (i.e., the system fails at the nth operation cycle, where $n \le W$), then a downtime cost C_d, and a purchasing cost C_p are incurred; also a refund amount $R(n)$ (see (1)) is gained due to the PRRW. Second, if the system fails after the warranty, but before the preventive replacement (i.e., the system fails at the nth operation cycle, where $W < n \le N$), then it incurs a downtime cost C_d, and a purchasing cost C_p, but without any gain due to the PRRW. Third, if the system does not fail before completing the Nth operation (i.e., the system fails at the nth operation cycle, where $n > N$), then a preventive replacement is performed with cost C_p, and the salvage value $v_s \cdot (n - N)$ is also gained from that un-failed system.

According to the above descriptions, the replacement cycle length, and the total cost in the renewal cycle (denoted by $T_1(N)$ and $C_1(N)$, respectively) become

$$T_1(N) = \begin{cases} n, & \text{if} \quad n \le W, \\ n, & \text{if} \quad W < n \le N, \\ N, & \text{if} \quad n > N, \end{cases} \tag{5}$$

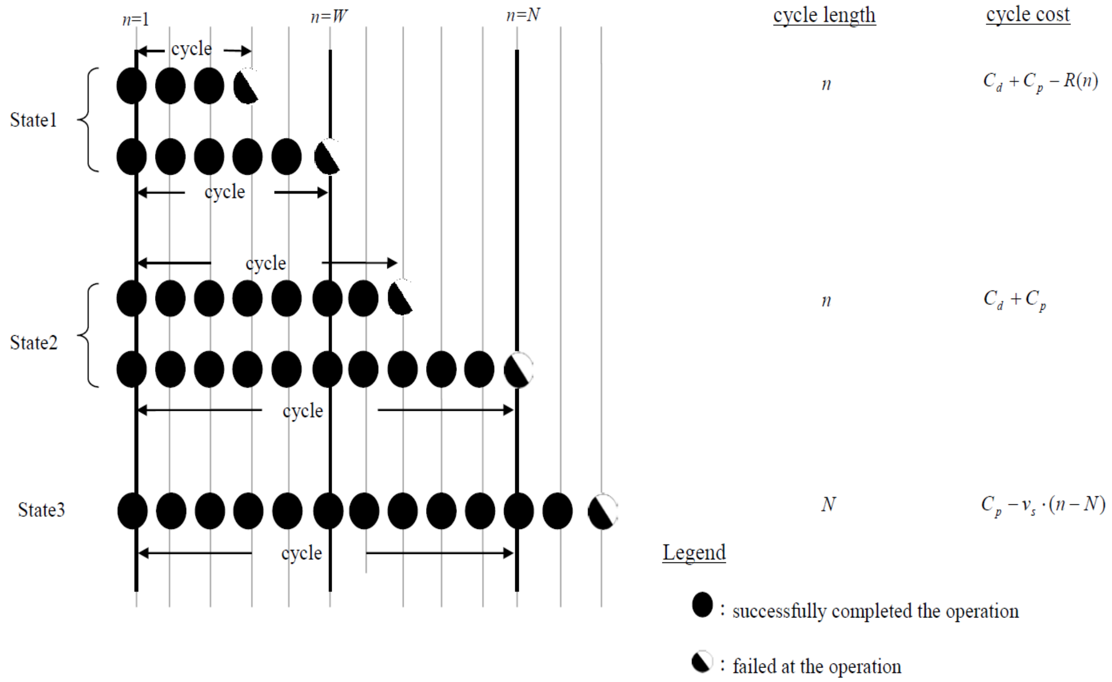

Figure 2. Possible replacements with PRRW when $N > W$.

and

$$C_1(N) = \begin{cases} C_d + C_p - R(n), & \text{if} \quad n \le W, \\ C_d + C_p, & \text{if} \quad W < n \le N, \\ C_p - v_s \cdot (n - N), & \text{if} \quad n > N. \end{cases} \tag{6}$$

Therefore, the long-run expected cost rate is

$$CR_1(N) = \frac{E[C_1(N)]}{E[T_1(N)]}$$

$$= \frac{\sum_{n=1}^{W}[C_d + C_p - R(n)]p_n + (C_d + C_p)\sum_{n=W+1}^{N} p_n + \sum_{n=N+1}^{\infty}[C_p - v_s(n-N)]p_n}{\sum_{n=1}^{W} n \cdot p_n + \sum_{n=W+1}^{N} n \cdot p_n + \sum_{n=N+1}^{\infty} N \cdot p_n}$$

$$= \frac{C_d \sum_{n=1}^{N} p_n + C_p \frac{\sum_{m=1}^{W} \sum_{n=m+1}^{\infty} p_n}{W} - v_s \sum_{m=N}^{\infty} \sum_{n=m+1}^{\infty} p_n}{\sum_{m=1}^{N} \sum_{n=m}^{\infty} p_n}. \tag{7}$$

Case 2. $N \le W$

When the operation cycle for preventive replacing a system is scheduled within the warranty period W, all the replacements (preventive or corrective) are performed within the warranty. However, it should be note that if a preventive replacement is performed at the completion of operation cycle N, no refund can be gained because such a replacement is scheduled, not resulting from failure. In this case, there exist two possible replacement states, as shown in Figure 3. First, if the system fails before the preventive replacement (i.e., the system fails at the nth operation cycle, where $n \le N \le W$), then a downtime cost C_d, and a purchasing cost C_p are incurred; and a refund amount $R(n)$ is also gained. Second, if the system does not fail before completing the Nth operation (i.e., the system fails at the nth operation cycle, where $n > N$), then a preventive replacement is performed at the completion of Nth operation with cost C_p, and the salvage value $v_s \cdot (n-N)$ is also gained.

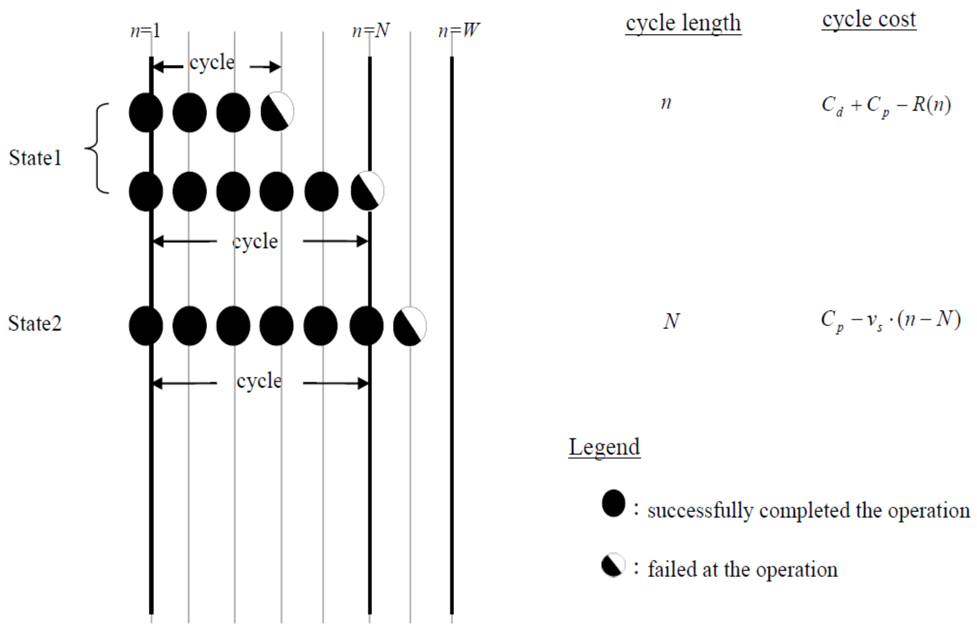

Figure 3. Possible replacements with PRRW when $N \le W$.

According to the above descriptions, the replacement cycle length, and the total cost in the renewal cycle (denoted by $T_2(N)$ and $C_2(N)$, respectively) are

$$T_2(N) = \begin{cases} n, & \text{if} \quad n \le N, \\ N, & \text{if} \quad n > N, \end{cases} \tag{8}$$

and

$$C_2(N) = \begin{cases} C_d + C_p - R(n), & \text{if} \quad n \le N, \\ C_p - v_s \cdot (n - N), & \text{if} \quad n > N. \end{cases} \tag{9}$$

Then the long-run expected cost rate becomes

$$\begin{aligned} CR_2(N) &= \frac{E[C_2(N)]}{E[T_2(N)]} \\ &= \frac{\sum_{n=1}^{N} [C_d + C_p - R(n)] p_n + \sum_{n=N+1}^{\infty} [C_p - v_s(n-N)] p_n}{\sum_{n=1}^{N} n \cdot p_n + N \cdot \sum_{n=N+1}^{\infty} p_n} \\ &= \frac{C_d \sum_{n=1}^{N} p_n + C_p \dfrac{(W-N) \sum_{n=N+1}^{\infty} p_n + \sum_{m=1}^{N} \sum_{n=m+1}^{\infty} p_n}{W} - v_s \sum_{m=N}^{\infty} \sum_{n=m+1}^{\infty} p_n}{\sum_{m=1}^{N} \sum_{n=m}^{\infty} p_n}. \end{aligned} \tag{10}$$

Also note that

$$CR_1(W) = CR_2(W) = \frac{C_d \sum_{n=1}^{W} p_n + C_p \dfrac{\sum_{m=1}^{W} \sum_{n=m+1}^{\infty} p_n}{W} - v_s \sum_{m=W}^{\infty} \sum_{n=m+1}^{\infty} p_n}{\sum_{m=1}^{W} \sum_{n=m}^{\infty} p_n} \tag{11}$$

from (7) and (10).

3. Optimal Policies

The main objective here is to derive the optimal number of operation cycles N_i^* for preventive replacement.

3.1. Optimal replacement policy without warranty

For a system without warranty, from (4), we see that the inequalities $CR_0(N+1) \geq CR_0(N)$ and $CR_0(N) < CR_0(N-1)$ hold iff

$$H(N) \geq \frac{C_p - v_s \cdot \mu}{C_d} \quad \text{and} \quad H(N-1) < \frac{C_p - v_s \cdot \mu}{C_d}, \tag{12}$$

where $H(N) = r_{N+1} \sum_{m=1}^{N} \sum_{n=m}^{\infty} p_n - \sum_{n=1}^{N} p_n$, which is just the same intermediate function that used in Chien [13]. Then, the following Lemma concerning the properties of $H(N)$ is summarized below, which is required and helpful to examine the existence and uniqueness of the optimal N_i^* .

Lemma 1. *Suppose that r_n is strictly increasing in n (i.e. IFR), then $H(n)$ is also strictly increasing in n. Furthermore, $\lim_{n\to 0} H(n) = H(0) = 0$ and $\lim_{n\to\infty} H(n) = H(\infty)$ $= r_\infty \mu - 1$.*

Proof. See the Appendix of [13] for the detailed proof.

Because most systems deteriorate due to the number of operations, the case that r_n has IFR will be focused throughout this paper. In this case, the optimal number of operation cycles N_0^*, for preventive replacing a system without warranty, can be easily obtained through (12), i.e., $H(N_0^*-1) < \left(C_p - v_s \cdot \mu\right) / C_d \leq H(N_0^*),$ and the property results are given in the following Theorem.

Theorem 1. *To consider salvage value of a system that operating in discrete time with an IFR r_n, the following results that concerning the optimal N_0^* are true.*

(i) When $C_p \leq v_s \cdot \mu,$ $N_0^* = 0.$

(ii) When $C_p > v_s \cdot \mu,$ if $H(\infty) > \left(C_p - v_s \mu\right) / C_d$, or equivalently $r_\infty > (C_p + C_d - v_s \cdot \mu)/(C_d \cdot \mu),$ then there exists a finite, unique N_0^* (i.e., $0 < N_0^* < \infty$) that satisfies the inequality

$$r_{N_0^*} \sum_{j=1}^{N_0^*-1} \sum_{i=j}^{\infty} p_i - \sum_{j=1}^{N_0^*-1} p_j < \frac{C_p - v_s \cdot \mu}{C_d} \leq r_{N_0^*+1} \sum_{j=1}^{N_0^*} \sum_{i=j}^{\infty} p_i - \sum_{j=1}^{N_0^*} p_j , \tag{13}$$

and the resulting expected cost rate satisfies the inequality

$$C_d \cdot r_{N_0^*} + v_s < CR_0\left(N_0^*\right) \leq C_d \cdot r_{N_0^*+1} + v_s . \tag{14}$$

Otherwise, $N_0^* = \infty$ and the resulting expected cost rate is $CR_0\left(N_0^*\right) = CR_0\left(\infty\right) =$

$\left(C_d + C_p\right)/\mu$.

Proof. (i) When $C_p \le v_s \cdot \mu$, and by (4), we have

$$CR_0(N+1) - CR_0(N) = \frac{C_d \cdot H(N) - \left(C_p - v_s \cdot \mu\right)}{\left(\sum_{m=1}^{N}\sum_{n=m}^{\infty} p_n\right)\left(\sum_{m=1}^{N+1}\sum_{n=m}^{\infty} p_n\right)} > 0\,. \tag{15}$$

Thus $CR_0(N)$ is strictly increasing in N, and thus $N_0^* = 0$.

(ii) When $C_p > v_s \cdot \mu$, then (12) is equivalent to $H(N-1) < \left(C_p - v_s \cdot \mu\right)/C_d \le H(N)$. And by Lemma 1, it is obvious that if $H(\infty) > \left(C_p - v_s\mu\right)/C_d$, or equivalently $r_\infty > \left(C_p + C_d - v_s \cdot \mu\right)/\left(C_d \cdot \mu\right)$, then there exists a finite, and unique N_0^* (i.e., $0 < N_0^* < \infty$) that satisfies $H(N_0^* - 1) < \left(C_p - v_s \cdot \mu\right)/C_d \le H(N_0^*)$, which is equivalent to (13); further through algebraic manipulation, the resulting expected cost rate satisfies (14). Otherwise, $N_0^* = \infty$ and $CR_0\left(N_0^*\right) = CR_0(\infty) = \left(C_d + C_p\right)/\mu$.

3.2. Optimal replacement policy with warranty

Again, to derive the optimal N_i^* under PRRW, the two cases have to be investigated separately: $N > W$ and $N \le W$, For $N > W$, let N_1^* be the optimal number of operation cycles for preventive replacement that minimize the cost rate $CR_1(N)$. Then, the following lemma concerning N_1^* can be obtained.

Lemma 2. *To consider salvage value for a system that operating in discrete time with an IFR r_n and under the PRRW with period W, the following results concerning the optimal N_1^* hold for $N = W+1, W+2, \cdots$.*

(i) When $C_p\left(\sum_{m=1}^{W}\sum_{n=m+1}^{\infty} p_n \big/ W\right) \le v_s \cdot \mu$, then $N_1^* = W+1$, and

$$CR_1\left(N_1^*\right) = CR_1(W+1) = \frac{C_d \cdot \sum_{n=1}^{W+1} p_n + C_p \cdot \dfrac{\sum_{m=1}^{W}\sum_{n=m+1}^{\infty} p_n}{W} - v_s \cdot \sum_{m=W+1}^{\infty}\sum_{n=m+1}^{\infty} p_n}{\sum_{m=1}^{W+1}\sum_{n=m}^{\infty} p_n}\,. \tag{16}$$

(ii) When $C_p\left(\sum_{m=1}^{W}\sum_{n=m+1}^{\infty} p_n \big/ W\right) > v_s \cdot \mu$,

(1) if $H(W) \ge [C_p\left(\sum_{m=1}^{W}\sum_{n=m+1}^{\infty} p_n \big/ W\right) - v_s \cdot \mu]/C_d$, then $N_1^* = W+1$, and $CR_1\left(N_1^*\right) = CR_1(W+1)$ is as given by (16).

(2) if $H(W) < [C_p\left(\sum_{m=1}^{W}\sum_{n=m+1}^{\infty} p_n \big/ W\right) - v_s \cdot \mu]\big/C_d < H(\infty)$*, then there exists a finite, unique* N_1^* *(i.e.,* $W < N_1^* < \infty$*), which satisfies the following inequality*

$$r_{N_1^*}\sum_{m=1}^{N_1^*-1}\sum_{n=m}^{\infty} p_n - \sum_{n=1}^{N_1^*-1} p_n < \frac{C_p \dfrac{\sum_{m=1}^{W}\sum_{n=m+1}^{\infty} p_n}{W} - v_s \cdot \mu}{C_d} \le r_{N_1^*+1}\sum_{m=1}^{N_1^*}\sum_{n=m}^{\infty} p_n - \sum_{n=1}^{N_1^*} p_n \,, \tag{17}$$

and the resulting expected cost rate satisfies

$$C_d \cdot r_{N_1^*} + v_s < CR_1\left(N_1^*\right) \le C_d \cdot r_{N_1^*+1} + v_s \,. \tag{18}$$

(3) if $H(\infty) \le [C_p\left(\sum_{m=1}^{W}\sum_{n=m+1}^{\infty} p_n \big/ W\right) - v_s \cdot \mu]\big/C_d$*, or equivalently*

$r_\infty \le \ [C_d + C_p\left(\sum_{m=1}^{W}\sum_{n=m+1}^{\infty} p_n \big/ W\right) - v_s \cdot \mu]\big/(\mu \cdot C_d)$*, then* $N_1^* = \infty$*, and*

$$CR_1\left(N_1^*\right) = CR_1(\infty) = \frac{C_d + C_p \dfrac{\sum_{m=1}^{W}\sum_{n=m+1}^{\infty} p_n}{W}}{\mu} \,. \tag{19}$$

Proof. (i) When $C_p\left(\sum_{m=1}^{W}\sum_{n=m+1}^{\infty} p_n \big/ W\right) \le v_s \cdot \mu$, and by (7), we have

$$CR_1(N+1) - CR_1(N) = \frac{C_d \cdot H(N) - \left(C_p \dfrac{\sum_{m=1}^{W}\sum_{n=m+1}^{\infty} p_n}{W} - v_s \cdot \mu\right)}{\left(\sum_{m=1}^{N}\sum_{n=m}^{\infty} p_n\right)\left(\sum_{m=1}^{N+1}\sum_{n=m}^{\infty} p_n\right)} > 0 \,. \tag{20}$$

Thus $CR_1(N)$ is strictly increasing in N ($> W$), and thus $N_1^* = W + 1$. Put $N = W + 1$ into (7), it yields (16).

(ii) When $C_p\left(\sum_{m=1}^{W}\sum_{n=m+1}^{\infty} p_n \big/ W\right) > v_s \cdot \mu$, then from (7), the inequalities $CR_1(N_1^* + 1) \ge CR_1(N_1^*)$ and $CR_1(N_1^*) < CR_1(N_1^* - 1)$ hold iff

$$H(N_1^*) \geq \frac{C_p \frac{\sum_{m=1}^{W}\sum_{n=m+1}^{\infty} p_n}{W} - v_s \cdot \mu}{C_d}, \text{ and } H(N_1^*-1) < \frac{C_p \frac{\sum_{m=1}^{W}\sum_{n=m+1}^{\infty} p_n}{W} - v_s \cdot \mu}{C_d}. \tag{21}$$

Thus, by (21) and Lemma 1,

(1) if $H(W) \geq [C_p\left(\sum_{m=1}^{W}\sum_{n=m+1}^{\infty} p_n \big/ W\right) - v_s \cdot \mu]\big/C_d$, then the optimal $N_1^* = W+1$, and corresponding cost rate $CR_1\left(N_1^*\right) = CR_1\left(W+1\right)$ is as given in (16).

(2) If $H(W) < [C_p\left(\sum_{m=1}^{W}\sum_{n=m+1}^{\infty} p_n \big/ W\right) - v_s \cdot \mu]\big/C_d$, then there exists a finite, unique N_1^* ($\geq W+1$), which satisfies (21) and is equivalent to the inequality (17). Further, through algebraic manipulation, the resulting expected cost rate $CR_1\left(N_1^*\right)$ satisfies (18).

(3) If $H(\infty) \leq [C_p\left(\sum_{m=1}^{W}\sum_{n=m+1}^{\infty} p_n \big/ W\right) - v_s \cdot \mu]\big/C_d$, which is equivalent to

$$r_\infty \leq \frac{C_d + C_p \frac{\sum_{m=1}^{W}\sum_{n=m+1}^{\infty} p_n}{W} - v_s \cdot \mu}{\mu \cdot C_d} \tag{22}$$

because $H(\infty) = r_\infty \mu - 1$. Then $N_1^* = \infty$, and the resulting expected cost rate is as given by (19).

Next, for $N \leq W$, let N_2^* be the optimal number of operation cycles for preventive replacement that minimize the cost rate $CR_2(N)$. Then, the following lemma concerning N_2^* can be obtained.

Lemma 3. *To consider salvage value for a system that operating in discrete time with an IFR r_n and under the PRRW with period W, if $\left[C_d - R(n)\right]r_n$ is strictly increasing in n, then the following results concerning the optimal N_2^* hold for $N = 1, 2, \cdots, W$.*

(i) When $C_p \leq v_s \cdot \mu$, $N_2^* = 0$.

(ii) When $C_p > v_s \cdot \mu$, the following two situations should be considered.

(1) For $C_p\left(\sum_{m=1}^{W}\sum_{n=m+1}^{\infty} p_n \big/ W\right) \leq v_s \cdot \mu$, there exists a unique N_2^* (i.e., $1 \leq N_2^* \leq W$) that minimize $CR_2(N)$, and the resulting expected cost rate satisfies

$$[C_d - R\left(N_2^*\right)]r_{N_2^*} + v_s < CR_2\left(N_2^*\right) \leq [C_d - R\left(N_2^*+1\right)]r_{N_2^*+1} + v_s, \tag{23}$$

where $R(\cdot)$ is defined by (1).

(2) For $C_p\left(\sum_{m=1}^{W}\sum_{n=m+1}^{\infty} p_n \big/ W\right) > v_s \cdot \mu$,

(a) if $H(W) > [C_p\left(\sum_{m=1}^{W}\sum_{n=m+1}^{\infty} p_n \big/ W\right) - v_s \cdot \mu] \big/ C_d$ *, then there exists a unique* N_2^* *(i.e.,* $1 \le N_2^* \le W$ *) that minimize* $CR_2(N)$*, and the resulting expected cost rate satisfies the inequality (23).*

(b) if $H(W) \le [C_p\left(\sum_{m=1}^{W}\sum_{n=m+1}^{\infty} p_n \big/ W\right) - v_s \cdot \mu] \big/ C_d$ *, then* $N_2^* = W$*, and* $CR_2(N_2^*) = CR_2(W)$ *is as given by (11).*

Proof. (i) By (10),

$$CR_2(N+1) - CR_2(N) = \frac{\Psi(N)}{\left(\sum_{m=1}^{N+1}\sum_{n=m}^{\infty} p_n\right)\left(\sum_{m=1}^{N}\sum_{n=m}^{\infty} p_n\right)}, \tag{24}$$

where

$$\Psi(N) = \left[C_d - R(N+1)\right]H(N) - C_p \frac{(W-N) + \sum_{m=1}^{N}\sum_{n=m+1}^{\infty} p_n}{W} + v_s \cdot \mu . \tag{25}$$

Since $\Psi(N+1) - \Psi(N) = \{[C_d - R(N+2)]r_{N+2} - [C_d - R(N+1)]r_{N+1}\} \times \sum_{m=1}^{N+1}\sum_{n=m}^{\infty} p_n > 0$ because $\left[C_d - R(n)\right]r_n$ is strictly increasing in n. Thus, $\Psi(N)$ is a strictly increasing function of N. Therefore, when $C_p \le v_s \cdot \mu$, then $\Psi(0) = -C_p + v_s \cdot \mu \ge 0$; that is $\Psi(N) \ge 0$ for all N, which implies that $CR_2(N)$ does not decrease in N. Hence, $N_2^* = 0$.

(ii) Further, from (10), the inequalities $CR_2(N+1) \ge CR_2(N)$ and $CR_2(N) < CR_2(N-1)$ hold iff $\Psi(N) \ge 0$ and $\Psi(N-1) < 0$. Therefore,

(1) when $C_p > v_s \cdot \mu$ and $C_p\left(\sum_{m=1}^{W}\sum_{n=m+1}^{\infty} p_n \big/ W\right) \le v_s \cdot \mu$, then we obtain $\Psi(0) = -C_p + v_s \cdot \mu < 0$ and $\Psi(W) = C_d \times H(W) - [C_p\left(\sum_{m=1}^{W}\sum_{n=m+1}^{\infty} p_n \big/ W\right) - v_s \cdot \mu] > 0$. Thus, there exists a unique N_2^* (i.e., $1 \le N_2^* \le W$) that satisfies $\Psi(N_2^*) \ge 0$ and $\Psi(N_2^* - 1) < 0$. Algebraic manipulation of $\Psi(N_2^* - 1) < 0 \le \Psi(N_2^*)$ yields the resulting expected cost rate satisfies (23).

On the other hand, (2) when $C_p > v_s \cdot \mu$, and $C_p\left(\sum_{m=1}^{W}\sum_{n=m+1}^{\infty} p_n \big/ W\right) > v_s \cdot \mu$, then

(a) if $H(W) > [C_p \left(\sum_{m=1}^{W} \sum_{n=m+1}^{\infty} p_n \middle/ W\right) - v_s \cdot \mu] \big/ C_d$, thus we have $\Psi(0) = -C_p + v_s \cdot \mu < 0$ and $\Psi(W) = C_d \cdot H(W) - [C_p \left(\sum_{m=1}^{W} \sum_{n=m+1}^{\infty} p_n \middle/ W\right) - v_s \cdot \mu] > 0$, this implies that there exists a unique N_2^* (i.e., $1 \le N_2^* \le W$) satisfies $\Psi(N_2^*) \ge 0$ and $\Psi(N_2^* - 1) < 0$, and the resulting expected cost rate satisfies (23). Otherwise, (b) if $H(W) \le [C_p \left(\sum_{m=1}^{W} \sum_{n=m+1}^{\infty} p_n \middle/ W\right) - v_s \cdot \mu] \big/ C_d$, $\Psi(W) \le 0$. Thus $N_2^* = W$ and $CR_2(N_2^*) = CR_2(W)$ is as given in (11).

It is worthy noting that the condition $C_p \le v_s \cdot \mu$ means that the purchasing cost of a new system is lower than the expected salvage value over its lifetime. That is, under this condition, the optimal replacement policy is always that the customer should preventively replace a new system when it is purchased. Theorem 1 and Lemma 3 confirm the state of affairs. Theorem 1 indicating that when $C_p \le v_s \cdot \mu$ is true for a system without warranty, the optimal number of operation cycles for preventive replacing a product is $N_0^* = 0$. Lemma 3 indicating that for a PRRW warranted system under the case $N \le W$, the optimal number of operation cycles for preventive replacement is $N_2^* = 0$. Furthermore, from Lemma 2, it shows that for a PRRW warranted system under the case $N > W$, the optimal operation cycles for preventive replacement is $N_1^* = W + 1$ because $C_p \left(\sum_{m=1}^{W} \sum_{n=m+1}^{\infty} p_n \middle/ W\right) < C_p \le v_s \cdot \mu$. In fact, however, it seems more reasonable that the sale price of a new system should be larger than its expected salvage value over its lifetime. Therefore, the following discussion on the optimal policies in the remainder of this paper will focus on the condition $C_p > v_s \cdot \mu$.

In the previous discussions of Lemmas 2 & 3, the local optimal replacement cycles for a PRRW warranted system in discrete time were derived under the constrain $N > W$ and $N \le W$. However, in practice, the preventive replacement timing should not be pre-determined to be in a certain interval. Therefore, it is important to investigate the global optimal replacement cycles N_W^* without any constraint. The global optimal number of operation cycles N_W^* for preventive replacement is defined as:

$$N_W^* = \begin{cases} N_1^*, & \text{if } CR_1\left(N_1^*\right) < CR_2\left(N_2^*\right), \\ N_2^*, & \text{if } CR_1\left(N_1^*\right) \ge CR_2\left(N_2^*\right). \end{cases} \tag{26}$$

Combining Lemmas 2 and 3, the following theorem concerning the N_W^* can be obtained.

Theorem 2. *To consider salvage value for a system that operating in discrete time with an IFR r_n and under the PRRW with period W, if $C_p > v_s \cdot \mu$, and $\left[C_d - R(n)\right]r_n$ is strictly increasing in n, then the following results hold.*

(i) For $C_p\left(\sum_{m=1}^{W}\sum_{n=m+1}^{\infty} p_n \big/ W\right) \le v_s \cdot \mu$, then $1 \le N_W^* \le W$, and the resulting expected cost rate satisfies the inequality (23).

(ii) For $C_p\left(\sum_{m=1}^{W}\sum_{n=m+1}^{\infty} p_n \big/ W\right) > v_s \cdot \mu$, then

(1) if $H(W) > [C_p\left(\sum_{m=1}^{W}\sum_{n=m+1}^{\infty} p_n \big/ W\right) - v_s \cdot \mu] \big/ C_d$, then $1 \le N_W^* \le W$ and the resulting expected cost rate satisfies the inequality (23).

(2) if $H(W) = [C_p\left(\sum_{m=1}^{W}\sum_{n=m+1}^{\infty} p_n \big/ W\right) - v_s \cdot \mu] \big/ C_d$, then $N_W^* = W$ and the resulting expected cost rate is given by (11).

(3) if $H(W) < [C_p\left(\sum_{m=1}^{W}\sum_{n=m+1}^{\infty} p_n \big/ W\right) - v_s \cdot \mu] \big/ C_d < H(\infty)$, then $W < N_W^* < \infty$, and the resulting expected cost rate satisfies the inequality (18).

(4) if $H(\infty) \le [C_p\left(\sum_{m=1}^{W}\sum_{n=m+1}^{\infty} p_n \big/ W\right) - v_s \cdot \mu] \big/ C_d$, then $N_W^* = \infty$ and the resulting expected cost rate is given by (19).

Based on Theorem 2, note that when the expected salvage value over the lifetime of a new system is larger than a threshold (i.e., $v_s \cdot \mu \ge C_p\left(\sum_{m=1}^{W}\sum_{n=m+1}^{\infty} p_n \big/ W\right)$), the system should be preventively replaced before the warranty expires, to take advantage of the salvage value. However, if $v_s \cdot \mu < C_p\left(\sum_{m=1}^{W}\sum_{n=m+1}^{\infty} p_n \big/ W\right)$, then the timing to perform a preventive replacement may be scheduled before or after the warranty expiration; the condition for whether a preventive replacement is performed within the warranty period or not depends on the relationship between the values $[C_p\left(\sum_{m=1}^{W}\sum_{n=m+1}^{\infty} p_n \big/ W\right) - v_s \cdot \mu] \big/ C_d$ and $H(W)$. Carefully checking the term $[C_p\left(\sum_{m=1}^{W}\sum_{n=m+1}^{\infty} p_n \big/ W\right) - v_s \cdot \mu] \big/ C_d$, we find that as the downtime cost C_d or the salvage value per cycle v_s become larger, then $H(W) \ge [C_p\left(\sum_{m=1}^{W}\sum_{n=m+1}^{\infty} p_n \big/ W\right) - v_s \cdot \mu] \big/ C_d$ becomes more likely and $[C_p\left(\sum_{m=1}^{W}\sum_{n=m+1}^{\infty} p_n \big/ W\right) - v_s \mu] \big/ C_d$ becomes smaller. Thus the optimal policy is that the system should be replaced preventively before the warranty expires to avoid system failures, or to take advantage of the salvage value of an un-failed system. Otherwise, the optimal timing for preventive replacement should be greater than the warranty period to take advantage of the warranty coverage. These properties are reasonable, and make sense.

4. Comparisons

In this section, the impact of a PRRW on the optimal discrete age-replacement policy is investigated by comparing the expected cost rates $CR_i(N)$ as well as the optimal number of operation cycles N_i^* for preventive replacement. First, we have the following corollary results.

Corollary 1. *$CR_0(N) > CR_1(N)$ for $N > W > 0$, and $CR_0(N) > CR_2(N)$ for any $0 < N \le W$.*

Proof. From (4), and (7), it is obviously that

$$CR_0(N) - CR_1(N) = \frac{C_p\left(1 - \frac{\sum_{m=1}^{W}\sum_{n=m+1}^{\infty} p_n}{W}\right)}{\sum_{m=1}^{N}\sum_{n=m}^{\infty} p_n} > 0 \tag{27}$$

for any $N > W > 0$.

And from (4), and (10),

$$CR_0(N) - CR_2(N) = \frac{C_p\left[\frac{(W-N)\sum_{n=1}^{N} p_n + \sum_{m=1}^{N}\sum_{n=1}^{m} p_n}{W}\right]}{\sum_{m=1}^{N}\sum_{n=m}^{\infty} p_n} > 0 \tag{28}$$

for any $0 < N \le W$.

Corollary 1 means that given any fixed number of operation cycles N for preventive replacement, the expected cost rate for a system without warranty is always greater than the expected cost rate for a system with PRRW. This results in turn implies that, when the optimal policies are attained for both cases (i.e., N_0^* and N_W^*), the optimal expected cost rate for a warranted system results in a smaller value.

Next, the difference between N_0^* and N_W^* is compared to show the effect PRRW. Through Theorems 1 and 2, $H(W)$ plays an important role in the comparison of N_0^* with N_W^*, and we have the following corollary results.

Corollary 2. *To consider salvage value for a system that operating in discrete time with an IFR* r_n *and under the PRRW with period* W*, if* $C_p > v_s \cdot \mu$ *and* $\left[C_d - R(n)\right]r_n$ *is strictly increasing in* n*, then the optimal* N_0^* *and* N_W^**, which minimize the long-run expected cost rate, have the following properties.*

(i) When $C_p\left(\sum_{m=1}^{W}\sum_{n=m+1}^{\infty} p_n \middle/ W\right) > v_s \cdot \mu$

(1) if $H(W) < [C_p\left(\sum_{m=1}^{W}\sum_{n=m+1}^{\infty} p_n \middle/ W\right) - v_s \cdot \mu]\big/ C_d$*, then* $W < N_W^* < N_0^*$.

(2) if $[C_p\left(\sum_{m=1}^{W}\sum_{n=m+1}^{\infty} p_n \middle/ W\right) - v_s \cdot \mu]\big/ C_d < H(W) < (C_p - v_s \cdot \mu)\big/ C_d$*, then* $0 < N_W^* \le W < N_0^*$.

(3) if $H(W) > \left(C_p - v_s \cdot \mu\right)\big/ C_d$*, then* $0 < N_W^* \le N_0^* \le W$ *or* $0 < N_0^* \le N_W^* \le W$.

(ii) When $C_p\left(\sum_{m=1}^{W}\sum_{n=m+1}^{\infty} p_n \middle/ W\right) \le v_s \cdot \mu$

(1) if $H(W) < \left(C_p - v_s \cdot \mu\right)\big/ C_d$*, then* $N_W^* \le W < N_0^*$.

(2) if $H(W) > \left(C_p - v_s \cdot \mu\right)\big/ C_d$*, then* $0 < N_W^* \le N_0^* \le W$ *or* $0 < N_0^* \le N_W^* \le W$.

Proof. By Theorem 1, the optimal N_0^* can be obtained by solving $H(N_0^* - 1) < \left(C_p - v_s \cdot \mu\right)\big/ C_d \le H\left(N_0^*\right)$. Because $H(n)$ is strictly increasing in n, thus if $H(W) < \left(C_p - v_s \cdot \mu\right)\big/ C_d$, then $N_0^* > W$; otherwise, $N_0^* \le W$.

(i) When $C_p\left(\sum_{m=1}^{W}\sum_{n=m+1}^{\infty} p_n \middle/ W\right) > v_s \mu$, because $[C_p\left(\sum_{m=1}^{W}\sum_{n=m+1}^{\infty} p_n \middle/ W\right) - v_s \cdot \mu]\big/ C_d$ $< \left(C_p - v_s \cdot \mu\right)\big/ C_d$ for any $W > 0$, thus we may divide the value of $H(W)$ into 3 regions. First, (1) if $H(W) < [C_p\left(\sum_{m=1}^{W}\sum_{n=m+1}^{\infty} p_n \middle/ W\right) - v_s \cdot \mu]\big/ C_d$, then $N_0^* > W$ and $N_W^* > W$ are hold by Theorems 1 and 2; and since $H(n)$ is strictly increasing in n, thus the result $W < N_W^* < N_0^*$ is true. Next, (2) if $[C_p\left(\sum_{m=1}^{W}\sum_{n=m+1}^{\infty} p_n \middle/ W\right) - v_s \cdot \mu]\big/ C_d$ $< H(W) < \left(C_p - v_s \cdot \mu\right)\big/ C_d$, then $N_0^* > W$ and $1 \le N_W^* \le W$ by Theorems 1 and 2, thus $0 < N_W^* \le W < N_0^*$ is true. Finally, (3) if $H(W) > \left(C_p - v_s \cdot \mu\right)\big/ C_d$, then $1 \le N_W^* \le W$ and $N_0^* \le W$ by Theorems 1 and 2; so it could be $0 < N_W^* \le N_0^* \le W$ or $0 < N_0^* \le N_W^* \le W$.

On the other hand, (ii) when $C_p\left(\sum_{m=1}^{W}\sum_{n=m+1}^{\infty} p_n \middle/ W\right) \le v_s \cdot \mu$, the Theorem 2 shows that N_W^* is always greater than or equal to W. By the similar way, we may divide the value of $H(W)$ into 2 regions. Thus, by Theorems 1 and 2, (1) if

$H(W) < (C_p - v_s \cdot \mu)/C_d$, then $N_W^* \le W < N_0^*$; (2) if $H(W) > (C_p - v_s \cdot \mu)/C_d$, then it could be $0 < N_W^* \le N_0^* \le W$ or $0 < N_0^* \le N_W^* \le W$.

To give a better illustration for the Corollary 2, Figures. 4 and 5 are provided to show the relationship between optimal N^* and $H(W)$. It indicating that adding a PRRW to a system not only reduces the long-run expected cost rate, but also effects the location of the optimal number of operation cycles for preventive replacement. More precisely, when the optimal N_0^* for a system without warranty is greater than W, a PRRW with period W will shorten the optimal N_W^* for preventive replacement. On the other hand, if the optimal N_0^* is less than W, then a PRRW with period W will also make the optimal N_W^* for preventive replacement within the warranty, but is may be that $N_W^* \le N_0^* \le W$ or $N_0^* \le N_W^* \le W$. Figure 6 is a combination of Figures 4 and 5 for the purpose of further illustration in a different perspective.

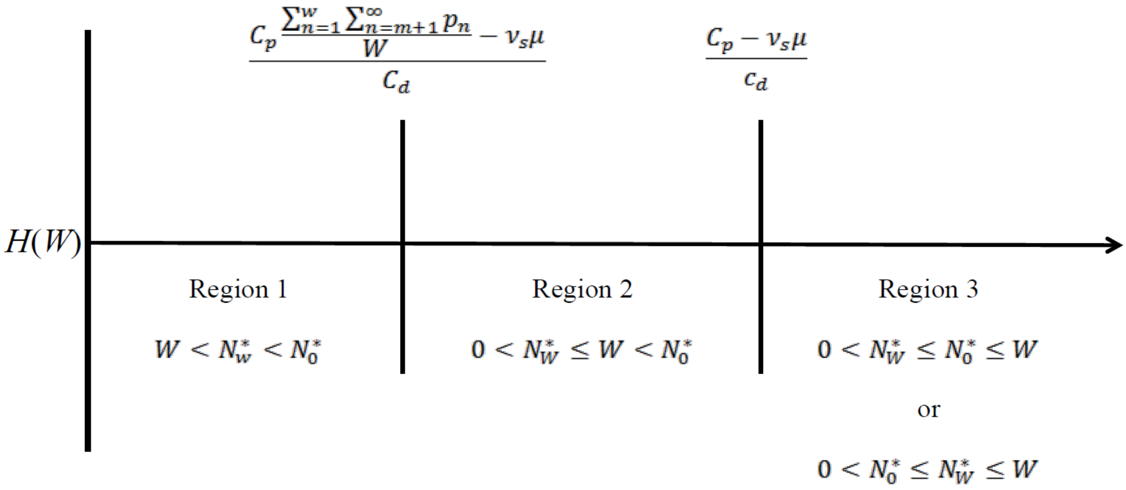

Figure 4. Relationship between optimal replacement ages (operation cycles) and $H(W)$, when $C_p(\sum_{m=1}^{W}\sum_{n=m+1}^{\infty} p_n/W) > v_s \cdot \mu$.

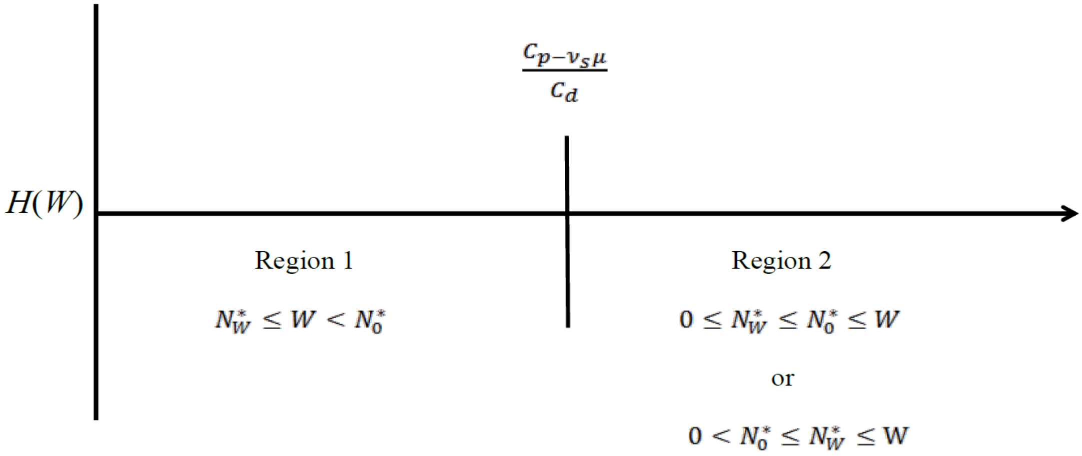

Figure 5. Relationship between optimal replacement ages (operation cycles) and $H(W)$, when $C_p(\sum_{m=1}^{W}\sum_{n=m+1}^{\infty} p_n/W) \le v_s \cdot \mu$

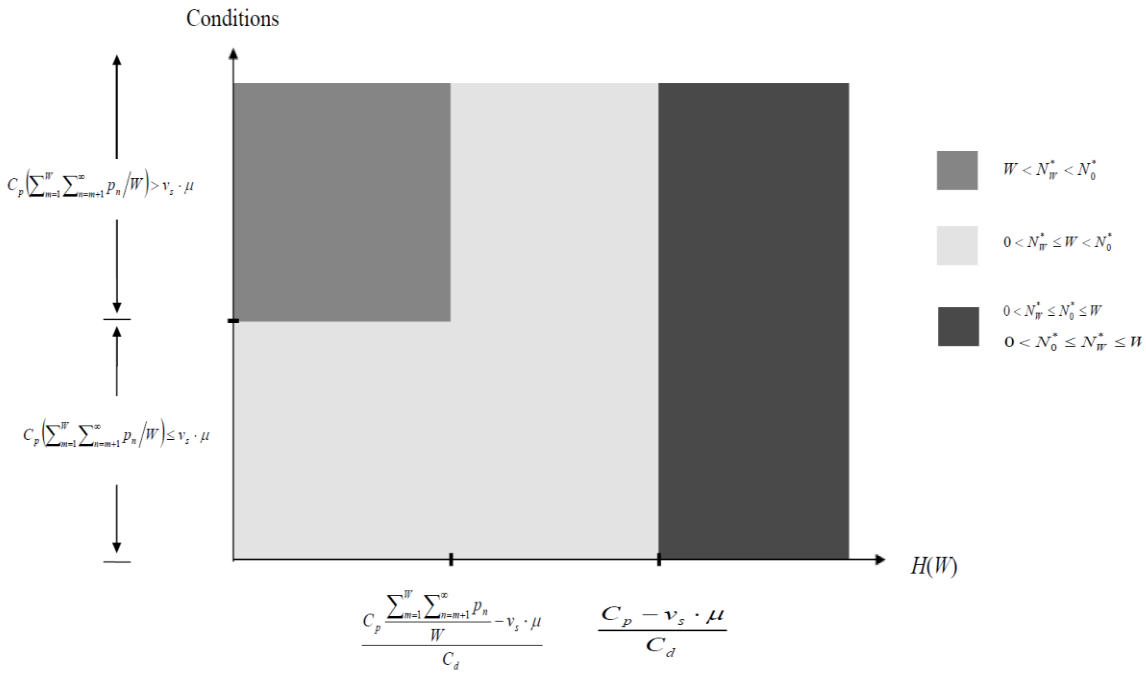

Figure 6. A diagram description for the Corollary 2.

Furthermore, the difference between the optimal cost rates provides a measure of the value of a PRRW. To study the variation in the magnitude of savings in the expected cost rate by PRRW, we can define

$$\Delta CR = \frac{CR_0\left(N_0^*\right) - CR\left(N_W^*\right)}{CR_0\left(N_0^*\right)}, \tag{29}$$

where $CR\left(N_W^*\right) = CR_1\left(N_1^*\right)$ when $N_W^* = N_1^*$, and $CR\left(N_W^*\right) = CR_2\left(N_2^*\right)$ when $N_W^* = N_2^*$. That is, ΔCR provides a measure of the value of PRRW, and it will be evaluated through a numerical example in the next Section.

5. A Numerical Example

This section investigates the sensitivity of the model parameters on the optimal discrete age replacement policy. Suppose that the failure distribution of a system operating in discrete time is a negative binomial one with a shape parameter of 2; that is,

$$p_n = np^2 q^{n-1}, \quad n = 1, 2, \cdots, \tag{30}$$

where $q = 1 - p$ $(0 < p < 1)$. Then, the mean number of operation cycles to failure is $\mu = (1+q)/p$; the failure rate is $r_n = np^2/(np+q)$, which is strictly increasing from p^2 to p; and the function $H(n)$ becomes $[(n+1)pq - q + q^{n+2}]/(np+1)$. Note that $H(0) = 0$, $H(\infty) = q$, and $H(n+1) - H(n) > 0$ for $n = 1, 2, 3, \cdots$ can be shown. Thus $H(n)$ is strictly increasing in n, and follows Lemma 1. This interesting discrete distribution was first introduced by Nakagawa and Osaki [24]. Nakagawa [23, p. 81] also applied this model as a discrete failure distribution when he discussed the replacement and maintenance policies.

Fix the warranty period $W = 20$, and the purchasing cost $C_p = 200$. The resulting optimal replacement policies, and corresponding expected cost rates, for both without warranty, and with a RFRW, are compared under various p, C_d, and v_s. The numerical calculation results are summarized in Table 1.

Table 1. Numerical results under various p, C_d, and v_s.

p	C_d	vs	N_0^*	$CR_0(N_0^*)$	N_1^*	$CR_1(N_1^*)$	N_2^*	$CR_2(N_2^*)$	N_W^*	$CR(N_W^*)$	ΔCR
		1	331	13.793	87	12.500	20	13.966	87	12.500	9.37%
		3	58	13.770	31	12.270	20	12.487	31	12.270	10.89%
	200	6	12	12.197	21	10.395	12	9.638	12	9.638	20.98%
		8	0	—	21	9.034	0	—	0	—	—
		10	0	—	21	7.673	0	—	0	—	—
		1	59	17.209	39	15.774	20	16.443	39	15.774	8.33%
		3	31	16.889	21	14.965	20	14.963	20	14.963	11.40%
	300	6	9	13.931	21	12.924	10	11.460	10	11.460	17.73%
		8	0	—	21	11.563	0	—	0	—	—
1/15		10	0	—	21	10.202	0	—	0	—	—
		1	28	23.338	22	21.382	20	21.396	22	21.382	8.38%
		3	18	22.143	21	20.022	16	19.721	16	19.721	10.93%
	500	6	6	16.791	21	17.981	7	14.563	7	14.563	13.26%
		8	0	—	21	16.620	0	—	0	—	—
		10	0	—	21	15.259	0	—	0	—	—
		1	14	35.442	21	34.026	14	32.896	14	32.896	7.18%
		3	11	32.348	21	32.665	11	29.806	11	29.806	7.85%
	1000	6	4	22.372	21	30.624	4	20.600	4	20.600	7.92%
		8	0	—	21	29.263	0	—	0	—	—
		10	0	—	21	27.902	0	—	0	—	—

p	C_d	vs	N_0^*	$CR_0(N_0^*)$	N_1^*	$CR_1(N_1^*)$	N_2^*	$CR_2(N_2^*)$	N_W^*	$CR(N_W^*)$	ΔCR
		1	654	17.391	62	15.178	20	16.052	62	15.178	12.72%
		3	69	17.388	28	14.969	20	15.076	28	14.969	13.91%
	200	6	20	16.897	21	13.709	15	13.369	15	13.369	20.87%
		8	7	14.549	21	12.820	9	11.078	9	11.078	23.85%
		10	0	—	21	11.931	0	—	0	—	—
		1	53	21.717	29	19.270	20	19.495	29	19.270	11.26%
		3	31	21.519	21	18.545	19	18.516	19	18.516	13.95%
	300	6	14	20.079	21	17.211	13	16.361	13	16.361	18.51%
		8	5	16.391	21	16.322	7	13.246	7	13.246	19.18%
1/12		10	0	—	21	15.433	0	—	0	—	—
		1	24	29.593	21	26.438	19	26.349	19	26.349	10.96%
		3	17	28.614	21	25.549	15	25.017	15	25.017	12.57%
	500	6	9	25.356	21	24.216	10	21.719	10	21.719	14.34%
		8	4	19.505	21	23.327	4	16.813	4	16.813	13.80%
		10	0	—	21	22.438	0	—	0	—	—
		1	12	45.288	21	43.950	12	41.558	12	41.558	8.23%
		3	9	42.548	21	43.061	10	38.881	10	38.881	8.61%
	1000	6	6	35.693	21	41.727	6	32.490	6	32.490	8.97%
		8	3	25.885	21	40.838	3	23.545	3	23.545	9.03%
		10	0	—	21	39.949	0	—	0	—	—

(—: undefined)

p	C_d	vs	N_0^*	$CR_0(N_0^*)$	N_1^*	$CR_1(N_1^*)$	N_2^*	$CR_2(N_2^*)$	N_W^*	$CR(N_W^*)$	ΔCR
		1	∞	21.052	45	17.681	20	18.132	45	17.681	16.01%
		3	83	21.052	23	17.448	20	17.467	23	17.448	17.11%
	200	6	26	20.881	21	16.558	16	16.298	16	16.298	21.94%
		8	13	20.062	21	15.959	12	15.037	12	15.037	25.04%
		10	4	16.724	21	15.360	6	12.559	6	12.559	24.90%
		1	48	26.300	23	22.566	20	22.588	23	22.566	14.19%
		3	30	26.171	21	21.974	18	21.889	18	21.889	16.36%
	300	6	16	25.294	21	21.076	14	20.342	14	20.342	19.57%
		8	9	23.599	21	20.478	10	18.619	10	18.619	21.10%
1/10		10	3	18.673	21	19.879	5	15.028	5	15.028	19.52%
		1	20	35.971	21	31.611	17	31.303	17	31.303	12.97%
		3	16	35.131	21	31.012	14	30.196	14	30.196	14.04%
	500	6	10	32.700	21	30.114	10	27.704	10	27.704	15.27%
		8	6	29.499	21	29.515	7	24.898	7	24.898	15.59%
		10	2	22.060	21	28.917	3	18.898	3	18.898	14.33%
		1	10	55.413	21	54.205	10	50.417	10	50.417	9.01%
		3	8	52.903	21	53.606	9	47.988	9	47.988	9.29%
	1000	6	6	47.248	21	52.708	6	42.818	6	42.818	9.37%
		8	4	41.112	21	52.110	4	37.339	4	37.339	9.17%
		10	2	29.095	21	51.511	2	26.371	2	26.371	9.36%

p	C_d	vs	N_0^*	$CR_0(N_0^*)$	N_1^*	$CR_1(N_1^*)$	N_2^*	$CR_2(N_2^*)$	N_W^*	$CR(N_W^*)$	ΔCR
		1	∞	26.666	29	21.160	20	21.254	29	21.160	20.64%
		3	124	26.666	21	20.885	20	20.869	20	20.869	21.73%
	200	6	33	26.638	21	20.375	16	20.133	16	20.133	24.42%
		8	19	26.407	21	20.035	14	19.412	14	19.412	26.48%
		10	11	25.602	21	19.695	11	18.372	11	18.372	28.23%
		1	43	33.325	21	27.313	19	27.266	19	27.266	18.18%
		3	29	33.255	21	26.973	17	26.772	17	26.772	19.49%
	300	6	17	32.778	21	26.463	14	25.732	14	25.732	21.49%
		8	12	31.943	21	26.123	12	24.720	12	24.720	22.61%
1/8		10	8	30.266	21	25.783	9	23.211	9	23.211	23.30%
		1	17	45.772	21	39.489	15	38.785	15	38.785	15.26%
		3	14	45.074	21	39.149	13	37.906	13	37.906	15.90%
	500	6	10	43.283	21	38.639	10	36.081	10	36.081	16.63%
		8	7	41.305	21	38.299	8	34.306	8	34.306	16.94%
		10	5	38.103	21	37.959	6	31.678	6	31.678	16.86%
		1	8	71.139	21	69.930	9	64.026	9	64.026	9.99%
		3	7	68.825	21	69.589	8	61.910	8	61.910	10.04%
	1000	6	6	64.222	21	69.079	6	57.689	6	57.689	10.17%
		8	4	59.967	21	68.739	5	53.825	5	53.825	10.24%
		10	3	53.808	21	68.399	4	48.531	4	48.531	9.80%

Below are a few points summarized on the basis of Table 1.

(1) Under the same failure distribution, both N_0^* and N_W^* decrease as C_d increases, or as v_s increases. This result is reasonable because a system with a higher downtime cost (or with a higher salvage value) should be replaced preventively more early to avoid failures (or to take advantage of the value of residual life). Moreover, it is also intuitive that $CR_i(N_i^*)$ ($i = 0,1,2$, and W) are increasing (decreasing) as C_d (v_s) increases. This result can be easily verified analytically through (4), (7), and (10).

(2) These numerical results are consistent with the characteristics identified in Theorems 1 & 2, and Lemmas 2 & 3. For example, when $p = 1/10$ or 1/8, and $C_d = 200$, v_s =1, the optimal N_0 calculation result is infinite (i.e., $N_0^* = \infty$) because the $H(\infty)$ calculated value is never larger than the $\left(C_p - v_s\mu\right)/C_d$ value; on the contrary, under the cooperating conditions of other parametric values, the condition of $H(\infty) > \left(C_p - v_s\mu\right)/C_d$ can be achieved, thus the N_0^* value will be finite (i.e., $N_0^* < \infty$), the details of this part were illustrated in Theorem 1. When $p = 1/15$ and $v_s = 8$ or 10, or when $p = 1/12$ and $v_s = 10$, the optimal N_0 and N_2 calculation results are 0 (i.e., $N_0^* = N_2^* = 0$) because the condition of $C_p \le v_s\mu$ is satisfied; on the contrary, in the case of other parametric values, C_p is bigger than $v_s\mu$, therefore both N_0^* and N_2^* are greater than or equal to 1 (that is $N_0^* \ge 1$, $N_2^* \ge 1$), with details as illustrated by Theorem 1 and Lemma 3. As for the calculation results of N_1^* and N_W^*, they are also consistent with the characteristics as described in Lemma 2 and Theorem 2, and thus are not elaborated here.

(3) By carefully observing these numerical calculation results, we also found that it confirms the advanced findings as proposed in Corollaries 1 & 2. Although the property proposed in Corollary 1 cannot be directly seen in Table I, however the authors found that $CR_0(N) > CR_2(N)$ for $1 \le N \le 20$ and $CR_0(N) > CR_1(N)$ for $N \ge 21$, these are all consistent the characteristics proposed in Corollary 1. Regarding the characteristics described in Corollary 2, they can be directly found in the Table I. For example, in the case of a certain fixed parameter combination, if the calculation value of N_0^* is greater than the warranty period W (i.e., $N_0^* > 20$), then the calculated value of N_W^* must be smaller than N_0^*, it may be $20 < N_W^* < N_0^*$ or $N_W^* \le 20 < N_0^*$; however, if the calculated value of N_0^* is less than or equal to W (i.e., $N_0^* \le 20$), then N_W^* is also less than or equal to W, but the relationship between N_W^* and N_0^* is uncertain, it can be $N_W^* \le N_0^* \le 20$ or $N_0^* \le N_W^* \le 20$. These are presented as shown in Table I. The above phenomena

and the properties mentioned in Corollary 2 are completely consistent.

(4) ΔCR, as shown in Table I, is defined by (29), and it represents the percentage of saved cost by using a system with warranty (i.e., PRRW), rather than a system without warranty. It can be found from Table I that, the cost saving percentage is 7.18% at least and can be up to 28.23%. Hence, we can find the benefits of operating a system in discrete time process with PRRW for the implementation of the optimal age replacement policy.

(5) We consider the case $p = 1/10$ (i.e., $\mu = 19$) for the purpose of verification the Figures 4 and 5. Fixed W, C_p and C_d at 20, 200 and 200, if $v_s = 6$, then it match the condition of Figure 4 (i.e., $136.23 \approx C_p\left(\sum_{m=1}^{W}\sum_{n=m+1}^{\infty} p_n \Big/ W\right) > v_s \cdot \mu = 114$, because $[C_p\left(\sum_{m=1}^{W}\sum_{n=m+1}^{\infty} p_n \Big/ W\right) - v_s \cdot \mu] / C_d \approx 0.111,\ (C_p - v_s \cdot \mu)/C_d = 0.43$ and $H(W) = 0.363,$ thus it also satisfies the condition of Region 2; further from Table 1, we observe that $N_0^* = 26$ and $N_W^* = 16$, thus it confirms the fact $0 < N_W^* \le W < N_0^*$. If $v_s = 10,$ then it match the condition of Figure 5 (i.e., $136.23 \approx C_p\left(\sum_{m=1}^{W}\sum_{n=m+1}^{\infty} p_n \Big/ W\right) \le v_s \cdot \mu = 190,$ because $(C_p - v_s \cdot \mu)/C_d = 0.05$ and $H(W) = 0.363,$ thus it also satisfies the condition of Region 2; further from Table 1, we observe that $N_0^* = 4$ and $N_W^* = 6,$ thus it confirms the fact $0 < N_0^* \le N_W^* \le W.$

6. Conclusion Remarks

Based on the phenomena observed from the above numerical calculation results as well as the technical analysis, we can further explain the connotations and summarize the practical information for the product users.

(1) When downtime cost C_d is bigger, no matter whether the system has a PRRW or not, its preventative replacement time should be as early as possible, as it can avoid the high price the system user has to pay due to unexpected breakdown. On the other hand, when system residual life's salvage value v_s is higher, no matter whether the system has a PRRW or not, its preventative replacement time should be also as early as possible, as it can allow the system user to enjoy the benefits of the salvage value of the preventively replaced system (since the product can operate normally because it has not been broken down).

(2) The optimal time of the preventive replacement of a system is subject to the

availability of the PRRW service of the system. When the calculation result of the optimal preventive replacement time of a system without warranty is out of the warranty period (i.e., $N_0^* > W$), then, if the system is changed to have the PRRW service, the optimal preventive replacement time N_W^* will become earlier, namely, $W < N_W^* < N_0^*$ or $N_W^* \le W < N_0^*$. When N_0^* calculation result is before the termination of the warranty period (i.e., $N_0^* \le W$), then N_W^* is also before the end of the warranty period (that is, $N_W^* \le N_0^* \le W$ or $N_0^* \le N_W^* \le W$). This suggests that, if the system has a PRRW service, then the optimal preventive replacement time will be close to the end of warranty period, or even within the warranty period, as it can enjoy the benefits of the PRRW service of the system.

(3) When system has a warranty service, this usually means the consumer has to bear more costs; that is, the price of buying a system with warranty is higher than the price of buying a system without warranty. ΔCR is just to measure the percentage of saved operating cost of using a system with warranty than a system without warranty. This can provide a reference to the system purchaser in deciding whether it is worth spending more money to buy a system with a warranty service.

The above messages are believed as considerably useful to the system user in determining whether to buy systems with PRRW service or not, as well as the implementation of the optimal age replacement policy.

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